Tableaux and the Asymmetric Simple Exclusion Process

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Dedications

To my beloved mentor

Amanda Himes

June 1, 1973 - March 9, 2013

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Abstract

Tableaux and the Asymmetric Simple Exclusion Process

Amanda Lohss

Various types of tableaux have recently been introduced due to a connection with the asymmetric simple exclusion process (ASEP) and have been the object of study in many recent papers. Relevant to this thesis, there have been several conjectures made regarding two such types of tableaux, namely staircase tableaux and tree–like tableaux. This thesis confirms these conjectures while proving other interesting results. More specifically, Hitczenko and Janson proved that distribution of symbols on the first diagonal of staircase tableaux is asymptotically normal, and they conjectured that other diagonals would be asymptotically Poisson. This thesis proves that conjecture for the kth diagonal where k is fixed. In addition, Laborde Zubieta gave a conjecture on the total number of corners in tree–like tableaux and the total number of corners in symmetric tree–like tableaux. Both conjectures are proven in this thesis. The proofs of these two conjectures are based on bijections with permutation tableaux and type–B permutation tableaux and consequently, results for these tableaux are also given. In addition, the limiting distributions of the number of occupied corners in tree–like tableaux and the number of diagonal boxes in symmetric tree–like tableaux are derived. These theorems extend results of Laborde-Zubieta and Aval et al. respectively.

Chapter 1: Introduction

A Young tableau is a left–aligned sequence of cells with weakly decreasing rows whose cells are filled with symbols taken from some alphabet. Young tableaux have been studied for over a hundred years, but new variations have recently been introduced [1], [10], [15], [30], [40] due to connections with various areas of physics, biology, and theoretical computer science. In particular, staircase tableaux (see Definition 1) were introduced ([15] and [16]) due to connections with the asymmetric simple exclusion process (ASEP) and Askey-Wilson polynomials. Similarly, tree–like tableaux (see Definition 2) were introduced (in [1]) due to connections with the partially asymmetric simple exclusion process (PASEP) and various other combinatorial objects. The principal motivation for studying these tableaux is the ASEP and its variations (such as the PASEP). See Section 1.3 for a description of the ASEP and its connection with tableaux.

This thesis confirms conjectures on both staircase tableaux and tree–like tableaux which are interesting due to the connection between tableaux and the ASEP. First, it was conjectured ([23]) that the distribution of the number of symbols along the kth diagonal of staircase tableaux follow a Poisson distribution for $2 \leq k \leq n$ as $n \to \infty$. This thesis verifies the conjecture for the kth diagonal where k is fixed. Second, it was conjectured (see Conjectures 4.1 and 4.2 in [31]) that the total number of corners in tree–like tableaux of size n is $n! \times \frac{n+4}{6}$ and the total number of corners in symmetric tree–like tableaux of size $2n+1$ is $2^n \times n! \times \frac{4n+13}{12}$. This thesis verifies these conjectures in their entirety. See Section 1.3 for the relevance of these conjectures in terms of the ASEP.

In addition to these conjectures, the distribution of boxes in staircase tableaux is computed and symbols on the kth diagonal where k is fixed are proven to be asymptotically independent (see Theorem 11 and Theorem 14). Other probabilistic properties of tree–like tableaux are also given. In particular, the number of occupied corners in a random tree–like tableau is proven to be asymptotically Poisson (Corollary 30), and the number of diagonal boxes in symmetric tree–like tableaux is proven to be asymptotically normal (Theorem 32). Most of the results on tree–like tableaux are based on bijections with permutation tableaux or type–B permutation tableaux (see Definitions 3 and 4). Permutation tableaux were introduced in [38] and type–B permutation tableaux were introduced in [33]. Due to this bijection, this thesis also provides new results on these tableaux as well (see Theorem 21 and Theorem 28).

This thesis is organized as follows. The necessary definitions and preceding results will be discussed in Section 1.1 as a background for this thesis. Section 1.3 contains a discussion on the ASEP and its significance to the results in this thesis. Section 1.3 is a short discussion on some of the combinatorial connections of staircase tableaux and tree–like tableaux. Chapter 2 contains the results on staircase tableaux which partially prove the conjecture by Hitczenko and Janson [23]. Chapter 3 is focused on tree-like tableaux, where Sections 3.2 and 3.5 contain the proofs of the conjectures by Laborde-Zubieta [31], Section 3.6.1 gives the distribution of the number of occupied corners in tree–like tableaux, and Section 3.7 gives the distribution of diagonal cells in symmetric tree–like tableaux.

1.1 Preliminaries

A *Ferrers diagram*, F, is a left–aligned sequence of cells with weakly decreasing rows. The *half–perimeter* of F is the number of rows plus the number of columns. The *border edges* of a Ferrers diagram are the edges of the southeast border, and the number of border edges is equal to the half–perimeter. Border edges are sometimes refereed to as a step (south or west). The rows and columns of a Ferrers diagram are enumerated beginning in the NW-corner. Let (i, j) refer to the cell (box) in the ith row and the jth column.

A *shifted Ferrers diagram* is a diagram obtained from a Ferrers diagram with k columns by adding k rows above it of lengths $k, (k-1), \ldots, 1$, respectively. The half–perimeter of the shifted Ferrers diagram is the same as the original Ferrers diagram (and similarly, the border edges are the same). The right–most cells of added rows are called *diagonal cells.*

A Young tableau is a Ferrers diagram whose cells are filled with symbols taken from some alphabet. The following Young tableaux will be discussed in this thesis:

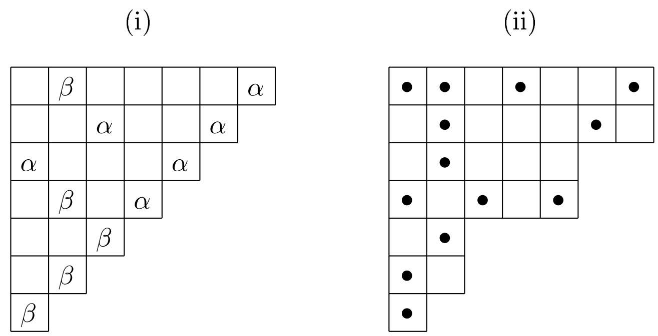


Figure 1.1: (i) A staircase tableau of size 7. (ii) A tree–like tableau size 13.

Definition 1. *A staircase tableau of size n is a Ferrers diagram of shape (n, n-1, ..., 1) such that:*

1. *The boxes are empty or contain an α, β, γ, or δ.*
2. *All boxes in the same column and above an α or γ are empty.*
3. *All boxes in the same row and to the left of an β or δ are empty.*
4. *Every box on the diagonal contains a symbol.*

Definition 2. *A tree–like tableau of size n is a Ferrers diagram of half-perimeter $n+1$ with some cells filled with a point (called pointed cells) according to the following rules:*

1. *The cell in the first column and first row is always pointed (this point is known as the root point).*
2. *Every row and every column contains at least one pointed cell.*
3. *For every pointed cell, all the cells above are empty or all the cells to the left are empty.*

Definition 3. *A permutation tableau of size n is a Ferrers diagram of half–perimeter n filled with 0's and 1's according to the following rules:*

1. *There is at least one 1 in every column.*
2. *There is no 0 with a 1 above it and a 1 to the left of it simultaneously.*

(i) (ii)

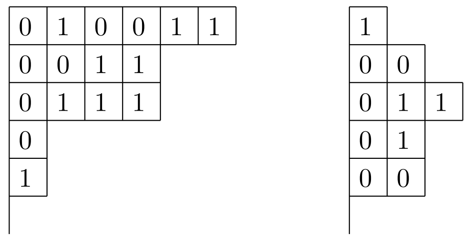


Figure 1.2: (i) A permutation tableau of size 12. (ii) A type-B permutation tableau of size 6.

Definition 4. *A type–B permutation tableau of size* n *is a shifted Ferrers diagram of half–perimeter* n *filled with* 0*'s and* 1*'s according to the following rules:*

1. *There is at least one* 1 *in every column.*
2. *There is no* 0 *with a* 1 *above it and a* 1 *to the left of it simultaneously.*
3. *If one of the diagonal cells contains a* 0 *(called a diagonal* 0*), then all the cells in that row are* 0*.*

Let $\mathcal{S}_n$ denote the set of all staircase tableaux of size n, $\mathcal{T}_n$ denote the set of all tree–like tableaux of size n, $\mathcal{P}_n$ denote the set of all permutation tableaux of size n, and $\mathcal{B}_n$ denote the set of all type–B permutation tableaux of size n. In addition to these tableaux, we are also interested in *symmetric tree–like tableaux*, a subset of tree–like tableaux which are symmetric about their main diagonal (see [1 Section 2.2] for more details). As noticed in [1], the size of a symmetric tree–like tableaux must be odd, and thus, we let $\mathcal{T}_{2n+1}^{sym}$ denote the set of all symmetric tree–like tableaux of size $2n+1$. It is known that $|\mathcal{S}_{n-1}| = |\mathcal{T}_n| = |\mathcal{P}_n| = n!$ and $|\mathcal{B}_n| = |\mathcal{T}_{2n+1}^{sym}| = 2^n n!$. Bijections between these objects will be discussed in Section 1.3.

Corners of a Ferrers diagram (or the associated tableau) are the cells in which both the right and bottom edges are border edges (i.e. a south step followed by a west step). In tree–like tableaux (symmetric or not) *occupied corners* are corners that contain a point. Diagonal boxes in symmetric tree–like tableaux refer to the boxes (i, i) which are always empty. Diagonals in staircase tableaux refer to the collection of boxes (i, j) such that $i + j = n - k + 2$ where $1 \le k \le n$.

In permutation tableaux and type–B permutation tableaux, a *restricted* 0 is a 0 which has a 1 above it in the same column. An *unrestricted row* is a row which does not contain any restricted 0's (and for type–B permutation tableaux, also does not contain a diagonal 0). Let $U_n(T)$ denote the number of unrestricted rows in a tableau T of size n. It is also convenient to denote a topmost 1 in a column by 1_T and a right-most restricted 0 by 0_R.

1.2 Connection to the ASEP

The underlying motivation for the conjectures proven in this thesis is the connection between tableaux and the asymmetric simple exclusion process (ASEP). The ASEP can be defined as a Markov chain, a stochastic process in which each state of the system only depends the previous state. More precisely, the ASEP is a Markov chain with n sites (n may approach infinity) where sites may be empty (denoted by a $\circ$) or occupied by a particle (denoted by a $\bullet$). Particles may only move to empty sites within the Markov chain. The rates in which particles may move are expressed as parameters. Particles may hop left with rate q and right with rate u. In addition, particles may enter the system with rates α and δ and exit with rates β and γ. Figure 1.3 gives an example of one particular state of the ASEP.

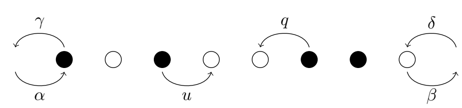


Figure 1.3: An example of the ASEP as defined by a Markov chain of size 8.

Various types of tableaux have been used to give a combinatorial formula for the steady state distribution of various forms of the ASEP such as staircase tableaux, tree–like tableaux, and permutation tableaux (all defined in Section 1.1) and also alternative tableaux [40], Catalan tableaux [34], [40], stammering tableaux [30], and rhombic staircase tableaux [10]. Each of these tableaux are associated with a state of the ASEP which is called the tableau's *type*.

In particular, staircase tableaux provide a formula for the steady state distribution of the ASEP with arbitrary parameters [15]. To do so, Definition 1 must be extended. After obtaining a staircase tableaux using Definition 1, the remaining empty boxes are then filled with u's and q's by the following rules:

1. Fill all empty boxes to the left of a β with a u.

2. Fill all empty boxes to the left of a δ with a q

3. Fill all empty boxes above an α with a u.

4. Fill all empty boxes above a γ with a q.

From this extended definition, each staircase tableau of size n is associated with a state of the ASEP with n sites (See Figure 1.4). This is done by aligning the Markov chain with the diagonal entries of the staircase tableau. A site is filled if the corresponding first diagonal entry is an α or a γ and a site is empty if the corresponding diagonal entry is a β or a δ. Each staircase tableau's associated state of the ASEP is called its type.

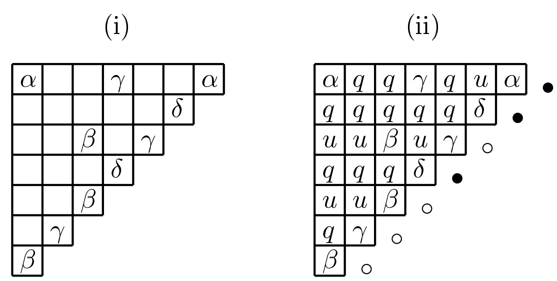


Figure 1.4: (i) A staircase tableau of size 7 with weight $\alpha^2\beta^3\delta^2\gamma^3$. (ii) The extension of (i) to a staircase tableau of weight $\alpha^2\beta^3\delta^2\gamma^3u^6q^{13}$ and type $\bullet\bullet\circ\bullet\circ\circ\circ$.

Using this association, it was shown in [16] that the steady state probability that the ASEP is in state η is given by:

$$\frac{\sum_{T\in\mathfrak{T}} wt(T)}{\sum_{S\in\mathcal{S}_n} wt(S)},$$

where $\mathfrak{T}$ is the set of all staircase tableau of type η. Due to this connection, the diagonals of staircase tableaux have been of interest and prompted the conjecture discussed in the introduction on the distribution of symbols on the kth diagonal.

Tree–like tableaux provide a combinatorial formula for the steady state distribution of the PASEP [31], [14], a variation of the ASEP where $\gamma = \delta = 0$ (i.e. particles may only enter the system from the left and exit from the right). The type of each tree–like tableau corresponds to the shape of its southeast border. In particular, each south step corresponds to a filled site and each west step corresponds to an empty site (ignoring the first and last step). See Figure 1.5.

Notice that a (outer) corner corresponds to a location in the PASEP where particles may jump to the right. The number of locations where particles may jump to the left (or *inner* corners of tree–like tableaux) is the number of outer corners minus 1. Therefore, corners in tree–like tableaux are clearly of interest in regards to the PASEP and is the motivation behind the conjecture discussed in the introduction.

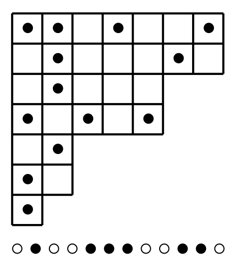


Figure 1.5: A tree–like tableau and its associated state of the PASEP as represented by a Markov chain of size 12.

1.3 Combinatorial Connections

Beyond their connection with the ASEP, staircase tableaux and tree–like tableaux have proven to be interesting in their own right as combinatorial objects and both have been the object of study in many recent papers (for example, [1], [12], [17], [23], and [31]). One interesting result is that of [12] and [16] which use staircase tableaux as a combinatorial formula for the moments of Askey-Wilson polynomials. In addition, tree–like tableaux are so named because they exhibit a natural tree-structure corresponding to binary trees (see Figure 1.6). There is a bijection between tree–like tableaux and permutations which preserves the binary tree associated with permutations (see [1]).

In addition, there exists bijections between staircase tableaux of size n, tree-like tableaux of size $n + 1$, and permutation tableaux of size $n + 1$ (these tableaux are also in bijection with alternative

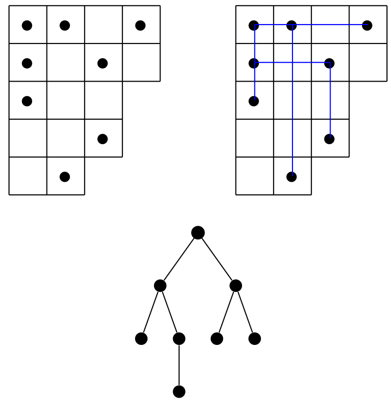


Figure 1.6: An example of a tree–like tableau and its associated binary tree.

tableaux which are not discussed here but see [23] for more details). There is also a bijection between symmetric tree–like tableaux and type–B permutation tableaux. For the purposes of this thesis, the bijection between (symmetric) tree-like tableaux and (type–B) permutation tableaux will be discussed further in Section 3.2 (Section 3.5). See [23] for more details on the other bijections.

Chapter 2: Staircase Tableaux

In this chapter, results on staircase tableaux diagonals will be presented. Prior to this research, the asymptotic distribution of parameters along the main diagonal were proven to be asymptotically normal, and the distribution of boxes along the main diagonal was given [23]. However, the distributions of parameters on other diagonals were not considered.

In [23], the distribution of each box in a staircase tableau was given, and the expected values of the number of symbols on the diagonals were computed. Based on those calculations, it was conjectured that the distribution of the number of symbols along the kth diagonal is asymptotically Poisson as k and the size of the tableau tend to infinity. This chapter will confirm this conjecture for finite k. More specifically, this chapter proves that the number of α's (resp. β's) along the kth diagonal follows the Poission distribution with parameter $1/2$ (Theorem 15) and that the symbols on the kth diagonal are asymptotically independent and thus, follow the Poisson distribution with parameter 1 (Theorem 17). In addition, the distribution of boxes along the kth diagonal is computed exactly for the second diagonal (Theorem 7) and asymptotically for the kth diagonal, where $k \geq 3$ is finite (Theorem 14).

2.1 Preliminaries

The formal definition for the kth diagonal of a staircase tableau is the collection of boxes (i, j) such that $i+j = n-k+2$. It is important to note that the kth diagonal in this thesis is the $(n-k+1)$th diagonal in [23]. It is convenient to enumerate diagonal boxes by their column position i.e. write $\{(n-k-l+2, l) : \ 1 \leq l \leq n-k+1\}$ for the boxes on the kth diagonal. Throughout this chapter, x_j^k will denote the event (or its indicator) that there is a symbol x on the kth diagonal in the j^{th} column of a staircase tableau; that is, there is an x in the $(n-k-j+2, j)$th box of the tableau.

For combinatorial considerations, all γ's can be replaced with α's and all δ's can be replaced with β's to obtain α/β-staircase tableaux. Recall that S_n is the set of all α/β-staircase tableaux.

The weight generating function for staircase tableaux is well-known and the following version for S_n is from [23],

$$Z_n(\alpha, \beta) := \sum_{S \in \mathcal{S}_n} wt(S) = \alpha^n \beta^n (a + b + n - 1)_n$$

where $wt(S)$ is the product of all symbols in S, $a := \alpha^{-1}$, $b := \beta^{-1}$ and $(x)_n = x(x-1)\dots(x-(n-1))$ is the falling factorial of x.

It is important to note that there is a simple involution between staircase tableaux of a particular size. If the rows and columns are transposed and the α's and β's are interchanged, a staircase tableau of the same size is obtained. This involution will allow the distribution of β's to be obtained from the distribution of α's.

A random weighted α/β-staircase tableaux was introduced in [23] in the following manner,

Definition 5. *For all $n \geq 1$, $\alpha, \beta \in [0, \infty)$ with $(\alpha, \beta) \neq (0, 0)$, define a random $S \in \mathcal{S}_n$ to be the α/β-staircase tableau chosen according to the following distribution,*

$$\mathbb{P}_{n,\alpha,\beta}(S) := \frac{wt(S)}{Z_n(\alpha, \beta)}, \quad S \in \mathcal{S}.$$

The parameters are allowed to approach ∞ by fixing α (resp. β) and taking the limit or by allowing $\alpha = \beta \to \infty$. Several such cases were considered in the literature (see [23 Section 3] for examples and discussion). As in [23] α and β are used in two different meanings, as fixed symbols in the tableaux and as the values of the parameters, but that should not cause any confusion. Note that any results for random α/β-staircase tableaux can be extended to random staircase tableaux with all four parameters, $\alpha, \gamma, \beta, \delta$. This can be done by randomly replacing each α with γ with probability $\frac{\gamma}{\alpha+\gamma}$ and each β with δ with probability $\frac{\delta}{\beta+\delta}$ independently for each occurrence.

Using the probability measure from Definition 5, the distribution of boxes was given in [23]. For boxes on the first diagonal ([23 Theorem 7.1]),

$$\mathbb{P}_{n,\alpha,\beta}(\alpha_j^1) = \frac{k + j + b - 2}{n + a + b - 1}, \quad \mathbb{P}_{n,\alpha,\beta}(\beta_j^1) = \frac{n - k - j + a + 1}{n + a + b - 1}. \tag{2.1}$$

For boxes on the kth diagonal, $k > 1$ ([23 Theorem 7.2]),

$$\mathbb{P}_{n,\alpha,\beta}(\alpha_j^k) = \frac{b+j-1}{(n-k+a+b+1)_2}, \qquad \mathbb{P}_{n,\alpha,\beta}(\beta_j^k) = \frac{n-k-j+a+1}{(n-k+a+b+1)_2}. \tag{2.2}$$

Of course boxes on the kth diagonal may be empty if $k > 1$, and this probability is the remainder of the above, i.e. $1 - \mathbb{P}_{n,\alpha,\beta}(\alpha_j^k) - \mathbb{P}_{n,\alpha,\beta}(\beta_j^k)$.

One more useful result from [23] relates to subtableaux. For arbitrary $S \in \mathcal{S}_n$, let $S[i,j]$ be the subtableau of S that begins in box (i,j), i.e. the first $i-1$ rows and $j-1$ columns of S are deleted. Then, as proven in [23 Theorem 6.1],

$$\mathbb{P}_{n,\alpha,\beta}(S) = \mathbb{P}_{n-i-j+2,\widehat{\alpha},\widehat{\beta}}(S[i,j]) \tag{2.3}$$

where $\widehat{\alpha}^{-1} = \alpha^{-1} + i - 1$ and $\widehat{\beta}^{-1} = \beta^{-1} + j - 1$.

The method of (factorial) moments will be used to prove convergence to a Poisson random variable. The following two lemmas will be useful for this method.

Lemma 1. *Let $Y = \sum_{j=1}^{m} I_j$, where (I_j) are indicator random variables. Then, for $r \geq 1$,*

$$\mathbb{E}(Y)_r = r! \sum_{1 \leq j_1 < \ldots < j_r \leq m} \mathbb{P}(I_{j_1} \cap \ldots \cap I_{j_r}),$$

where $\mathbb{E}(X)_r = \mathbb{E}[X(X-1)\ldots(X-(r-1))]$ is the r^{th} factorial moment.

Proof. Notice that

$$\begin{aligned}
z^Y &= z^{\sum_{j=1}^m I_j} = \prod_{j=1}^m z^{I_j} = \prod_{j=1}^m (1+(z-1))^{I_j} = \prod_{j=1}^m (1 + I_j(z-1)) \\
&= 1 + \sum_{r=1}^m \left(\sum_{1 \leq j_1 < \ldots < j_r \leq m} \left(\prod_{k=1}^r I_{j_k} \right) \right) (z-1)^r \\
&= 1 + \sum_{r=1}^m (z-1)^r \left(\sum_{1 \leq j_1 < \ldots < j_r \leq m} \left(\prod_{k=1}^r I_{j_k} \right) \right).
\end{aligned}$$

Thus,

$$\mathbb{E}(z^Y) = 1 + \sum_{r=1}^{m} (z-1)^r \left(\sum_{1 \le j_1 < \ldots < j_r \le m} \mathbb{P}(I_{j_1} \cap \ldots \cap I_{j_r}) \right).$$

Hence

$$\mathbb{E}(Y)_r = \frac{d^r}{dz^r} (\mathbb{E} z^Y)|_{z=1} = r! \left(\sum_{1 \le j_1 < \ldots < j_r \le m} \mathbb{P}(I_{j_1} \cap \ldots \cap I_{j_r}) \right).$$

□

Lemma 2. *Let*

$$J_{r,m} := \{1 \le j_1 < \ldots < j_r \le m : \; j_k \le j_{k+1} - 2, \;\; \forall k = 1, 2, \ldots, r-1\}.$$

Then

$$\sum J_{r,m} \left(\prod_{k=1}^{r} j_{r-k+1} \right) = \frac{(m+1)_{2r}}{2^r r!}.$$

Proof. By induction on r. When $r = 1$:

$$\sum_{J_{1,m}} \left(\prod_{k=1}^{1} j_{1-k+1} \right) = \sum_{j_1=1}^{m} j_1 = \frac{(m+1)m}{2}.$$

Assume the statement holds for $r-1$. Then:

$$\begin{aligned}
\sum_{J_{r,m}} \left(\prod_{k=1}^{r} j_{r-k+1} \right) &= \sum_{j_r=2r-1}^{m} j_r \left(\sum_{J_{r-1, j_r-2}} \prod_{k=2}^{r} j_{r-k+1} \right) \\
&= \sum_{j_r=2r-1}^{m} j_r \frac{(j_r-1)_{2(r-1)}}{2^{r-1}(r-1)!} \\
&= \frac{1}{2^{r-1}(r-1)!} \sum_{j_r=2r-1}^{m} (j_r)_{2r-1}
\end{aligned}$$

where the second equality is by the induction hypothesis. Since

$$\sum_{j_r=2r-1}^{m} (j_r)_{2r-1} = \sum_{j_r=0}^{m} (j_r)_{2r-1}$$

the lemma will be proved once it is verified that

$$\sum_{j=0}^{m}(j)_t = \frac{(m+1)_{t+1}}{t+1},$$

for any non-negative integer t (and apply it with $t = 2r-1$). Using the identity

$$\sum_{j=0}^{m}\binom{j}{t} = \binom{m+1}{t+1}$$

(see, e.g. [22 Formula (5.10)]) notice that

$$\sum_{j=0}^{m}(j)_t = \sum_{j=0}^{m}\frac{j!}{(j-t)!} = t!\sum_{j=0}^{m}\binom{j}{t} = t!\binom{m+1}{t+1} = t!\frac{(m+1)_{t+1}}{(t+1)!} = \frac{(m+1)_{t+1}}{m+1},$$

as asserted. $\square$

To apply the preceding lemmas, the joint distribution of boxes on the diagonal must be computed. The exact formulas for the second diagonal will be computed (see Theorems 5 and 7) while asymptotic formulas for the kth diagonal will be computed (see Theorems 11 and 14 below). It should be noted that the joint distribution of symbols for the main diagonal was given in [23 Theorem 7.6], but since boxes on the main diagonal cannot be empty it is a different issue.

2.2 Boxes on the Second Diagonal

In this section, the distribution of boxes on the 2nd diagonal will be computed exactly. First, consider the following two lemmas which give the probability of an arbitrary staircase tableau in $\mathcal{S}_n$ that is conditioned on having an α or a β in the box $(n-1,1)$. The statements follow almost immediately from the definition of staircase tableaux, but will be important in proving the results.

Lemma 3. *If $S \in \mathcal{S}_{n,\alpha,\beta}$ is conditioned on having α in box $(n-1,1)$, then the subtableau $S_{n,\alpha,\beta}[1,3] \in \mathcal{S}_{n-2,\alpha,\beta}$. In other words, for $S \in \mathcal{S}_{n,\alpha,\beta}$*

$$\mathbb{P}_{n,\alpha,\beta}(S \mid \alpha_{n-1,1}) = \mathbb{P}_{n-2,\alpha,\beta}(S[1,3]).$$

Proof. If $S \in \mathcal{S}_n$ is a staircase tableau such that $\alpha_{n-1,1}$, then $\beta_{n,1}$ and $\alpha_{n-1,2}$ by the rules of staircase tableaux. The first and second column are otherwise empty by those same rules. The remainder, $S[1,3]$, is an arbitrary staircase tableau of size $n-2$. Therefore, the lemma follows. $\square$

Lemma 4. *Let $(S)_{i,j}$ be a subtableau of S with the ith row and the jth column removed. If $S \in \mathcal{S}_{n,\alpha,\beta}$ is conditioned on having β in box $(n-1,1)$, then the subtableau $(S)_{n-1,2}$ is random tableau in $\mathcal{S}_{n-1,\alpha,\beta}$ conditioned on having a β in the $(n-1,1)$ box. In other words*

$$\mathbb{P}_{n,\alpha,\beta}(S \mid \beta_{n-1,1}) = \mathbb{P}_{n-1,\alpha,\beta}((S)_{n-1,2} \mid \beta_{n-1,1}).$$

Proof. If $S \in \mathcal{S}_n$ is a staircase tableau such that $\beta_{n-1,1}$, then $\alpha_{n-1,2}$ and $\beta_{n,1}$ by the rules of staircase tableaux. The second column is otherwise empty by those same rules. The nth row only has one box, $(n,1)$, which must be a β. The remainder is an arbitrary staircase tableau of size $n-1$ conditioned to have a β in box $(n-1,1)$. Therefore, the lemma follows. $\square$

The goal is to completely describe the distribution of fixed boxes along the second diagonal. First, the following is the distribution of fixed boxes which contain an α. Note that the superscript 2 to indicate a symbol on the 2nd diagonal will be omitted for the rest of this section since only the 2nd diagonal will be considered.

Theorem 5. *Let $1 \leq j_1 < ... < j_r \leq n-1$. If*

$$j_k \leq j_{k+1} - 2, \ \ \forall k = 1, 2, ..., r-1 \tag{2.4}$$

then

$$\mathbb{P}_{n,\alpha,\beta}(\alpha_{j_1}, \ldots, \alpha_{j_r}) = \prod_{k=1}^{r} \frac{b + j_{r-k+1} - 2r + 2k - 1}{(n + a + b - 2r + 2k - 1)_2}.$$

(For $r = 1$, this is (2.2)). Otherwise,

$$\mathbb{P}_{n,\alpha,\beta}(\alpha_{j_1}, \ldots, \alpha_{j_r}) = 0.$$

Proof. First note that when (2.4) fails there exists j_k such that $j_k = j_{k+1} - 1$ and thus there must be two α's in boxes side by side on the $(n-i,i)$ diagonal. But this is impossible by the rules of staircase tableaux as no symbol can be put in the diagonal box $(n-j_k, j_{k+1})$ adjacent to these two boxes. Therefore the probability is 0.

Suppose now that (2.4) holds and proceed by induction on r. Set

$$\widehat{\beta}^{-1} = \beta^{-1} + j_1 - 1, \quad \widehat{n} := n - (j_1 - 1), \quad \widehat{j}_l := j_l - (j_1 - 1), \quad 1 \le l \le r. \tag{2.5}$$

Then by (2.3)

$$\mathbb{P}_{n,\alpha,\beta}(\alpha_{j_1}, \ldots, \alpha_{j_r}) = \mathbb{P}_{\widehat{n},\alpha,\widehat{\beta}}(\alpha_{\widehat{j}_1}, \ldots, \alpha_{\widehat{j}_r}) = \mathbb{P}_{\widehat{n},\alpha,\widehat{\beta}}(\alpha_{\widehat{j}_2}, \ldots, \alpha_{\widehat{j}_r} | \alpha_{\widehat{j}_1}) \cdot \mathbb{P}_{\widehat{n},\alpha,\widehat{\beta}}(\alpha_{\widehat{j}_1}).$$

Since $\widehat{j}_1 = 1$, by Lemma 3 and the induction hypothesis (applied with $\widetilde{n} := \widehat{n} - 2$, $\widetilde{j}_l := \widehat{j}_{l+1} - 2$, $1 \le l \le r-1$)

$$\begin{aligned} \mathbb{P}_{\widehat{n},\alpha,\widehat{\beta}}(\alpha_{\widehat{j}_2}, \ldots, \alpha_{\widehat{j}_r} \mid \widehat{\alpha}_{j_1}) \quad = \mathbb{P}_{\widehat{n}-2,\alpha,\widehat{\beta}}(\alpha_{\widehat{j}_2 - 2}, \ldots, \alpha_{\widehat{j}_r - 2}) \quad &= \prod_{k=1}^{r-1} \frac{\widehat{b} + \widetilde{j}_{(r-1)-k+1} - 2(r-1) + 2k - 1}{(\widetilde{n} + a + \widehat{b} - 2(r-1) + 2k - 1)_2} \\ &= \prod_{k=1}^{r-1} \frac{b + j_{r-k+1} - 2r + 2k - 1}{(n + a + b - 2r + 2k - 1)_2}, \end{aligned}$$

where, in the last step (2.5) was applied. By (2.2) and (2.3):

$$\mathbb{P}_{\widehat{n},\alpha,\widehat{\beta}}(\alpha_1) = \frac{\widehat{b}}{(\widehat{n} + a + \widehat{b} - 1)_2} = \frac{b + j_1 - 1}{(n + a + b - 1)_2}.$$

Therefore,

$$\mathbb{P}_{n,\alpha,\beta}(\alpha_{j_1}, \ldots, \alpha_{j_r}) \quad = \quad \prod_{k=1}^{r} \frac{b + j_{r-k+1} - 2r + 2k - 1}{(n + a + b - 2r + 2k - 1)_2}$$

which proves the result. $\square$

Corollary 6. *Let* $1 \leq j_1 < ... < j_r \leq n-1$. *If*

$$j_k \leq j_{k+1} - 2, \ \ \forall k = 1, 2, ..., r-1 \tag{2.6}$$

then

$$\mathbb{P}_{n,\alpha,\beta}(\beta_{j_1}, \ldots, \beta_{j_r}) = \prod_{k=1}^{r} \frac{n - j_{r-k+1} + a - 2r + 2k - 1}{(n + a + b - 2r + 2k - 1)_2}.$$

Otherwise,

$$\mathbb{P}_{n,\alpha,\beta}(\alpha_{j_1}, \ldots, \alpha_{j_r}) = 0.$$

Proof. This follows from the involution discussed in Section 2.1. □

The second diagonal may have empty boxes. Therefore, in order to completely describe the distribution of fixed boxes on the second diagonal, α and β must be considered collectively. The following is the distribution of non-empty boxes along the second diagonal.

Theorem 7. *Let* $1 \leq j_1 < ... < j_r \leq n-1$. *If (2.4) holds then*

$$\mathbb{P}_{n,\alpha,\beta}(x_{j_1}, \ldots, x_{j_r}) = \prod_{k=1}^{r} \frac{1}{n + a + b - r + k - 1}.$$

(For $r = 1$*, this is obtained by adding the expressions in (2.2).) Otherwise,*

$$\mathbb{P}_{n,\alpha,\beta}(x_{j_1}, \ldots, x_{j_r}) = 0.$$

Proof. Suppose (2.4) holds and proceed by induction on r. As in the proof of Theorem 5 by passing to $\widehat{n} := n - (j_1 - 1)$, $\widehat{j}_i := j_i - (j_1 - 1)$, and $\widehat{b} := b + j_1 - 1$ assume that $j_1 = 1$.

By the law of total probability,

$$\begin{aligned}\mathbb{P}_{n,\alpha,\beta}(x_1, \ldots, x_{j_r}) \quad &= \quad \mathbb{P}_{n,\alpha,\beta}(x_{j_2}, \ldots, x_{j_r} \mid x_1 = \alpha)\mathbb{P}_{n,\alpha,\beta}(x_1 = \alpha) \\ &\quad + \mathbb{P}_{n,\alpha,\beta}(x_{j_2}, \ldots, x_{j_r} \mid x_1 = \beta)\mathbb{P}_{n,\alpha,\beta}(x_1 = \beta).\end{aligned}$$

Now consider two cases:

Case 1: $x_1 = \alpha$. By Lemma 3 and the induction hypothesis:

$$\begin{aligned}
\mathbb{P}_{n,\alpha,\beta}(x_{j_2}, \ldots, x_{j_r} \mid x_1 = \alpha) \qquad &= \mathbb{P}_{n-2,\alpha,\beta}(x_{j_2-2}, \ldots, x_{j_r-2}) \\
&= \prod_{k=1}^{r-1} \frac{1}{n-2+a+b-(r-1)+k-1} \\
&= \prod_{k=1}^{r-1} \frac{1}{n+a+b-r+k-2}
\end{aligned}$$

and by (2.2),

$$\mathbb{P}_{n,\alpha,\beta}(x_1 = \alpha) = \frac{b}{(n+a+b-1)_2}.$$

Therefore,

$$\mathbb{P}_{n,\alpha,\beta}(\alpha_1, x_{j_2}, \ldots, x_{j_r}) = \frac{b}{(n+a+b-1)_2} \cdot \prod_{k=1}^{r-1} \frac{1}{n+a+b-r+k-2}.$$

Case 2: $x_1 = \beta$. By Lemma 4

$$\begin{aligned}
\mathbb{P}_{n,\alpha,\beta}(x_{j_2}, \ldots, x_{j_r} \mid x_{j_1} = \beta) \quad &= \mathbb{P}_{n-1,\alpha,\beta}(x_{j_2-1}, \ldots, x_{j_r-1} \mid \beta_{n-1,1}) \\
&= \frac{\mathbb{P}_{n-1,\alpha,\beta}(x_{j_2-1}, \ldots, x_{j_r-1}, \beta_{n-1,1})}{\mathbb{P}_{n-1,\alpha,\beta}(\beta_{n-1,1})}.
\end{aligned}$$

The numerator is equal to

$$\begin{aligned}
&\mathbb{P}_{n-1,\alpha,\beta}(x_{j_2-1}, \ldots, x_{j_r-1}) - \mathbb{P}_{n-1,\alpha,\beta}(x_{j_2-1}, \ldots, x_{j_r-1},\ \alpha_{n-1,1}) \\
&\qquad = \mathbb{P}_{n-1,\alpha,\beta}(x_{j_2-1}, \ldots, x_{j_r-1}) - \mathbb{P}_{n-1,\alpha,\beta}(x_{j_2-1}, \ldots, x_{j_r-1} \mid \alpha_{n-1,1})\mathbb{P}_{n-1,\alpha,\beta}(\alpha_{n-1,1}).
\end{aligned} \tag{2.7}$$

By [23 Lemma 7.5] and the induction hypothesis the conditional probability above is

$$\mathbb{P}_{n-2,\alpha,\beta}(x_{j_2-2}, \ldots, x_{j_r-2}) = \prod_{k=1}^{r-1} \frac{1}{n+a+b-r+k-2}. \tag{2.8}$$

By (2.2), (2.1), and the induction hypothesis,

$$\mathbb{P}_{n,\alpha,\beta}(x_1=\beta)=\frac{n+a-2}{(n+a+b-1)_2} \tag{2.9}$$

$$\frac{1}{\mathbb{P}_{n-1,\alpha,\beta}(\beta_{n-1,1})}=\frac{n+a+b-2}{n+a-2} \tag{2.10}$$

$$\mathbb{P}_{n-1,\alpha,\beta}(x_{j_2-1},\ldots,x_{j_r-1})=\prod_{k=1}^{r-1}\frac{1}{n-1+a+b-r+k} \tag{2.11}$$

$$\mathbb{P}_{n-1,\alpha,\beta}(\alpha_{n-1,1})=\frac{b}{n+a+b-2}. \tag{2.12}$$

Combining (2.8)–(2.12):

$$\begin{aligned}\mathbb{P}_{n,\alpha,\beta}(\beta_1,x_{j_2},\ldots,x_{j_r})=\frac{1}{n+a+b-1}\cdot\Big(\prod_{k=1}^{r-1}&\frac{1}{n-1+a+b-r+k}\\ &-\frac{b}{n+a+b-2}\prod_{k=1}^{r-1}\frac{1}{n+a+b-r+k-2}\Big).\end{aligned}$$

Adding Case 1 and Case 2:

$$\begin{aligned}\mathbb{P}_{n,\alpha,\beta}(x_1,...,x_{j_r}) &= \frac{1}{n+a+b-1}\cdot\prod_{k=1}^{r-1}\frac{1}{n+a+b-r+k-1}\\ &= \prod_{k=1}^{r}\frac{1}{n+a+b-r+k-1}\end{aligned}$$

which proves our assertion when (2.4) holds.

If there exists j_k such that $j_k=j_{k+1}-1$, then $\{x_{j_1},\ldots,x_{j_r}\}$ implies that two boxes side by side on the $(n-i,i)$ diagonal are non-empty, which is impossible by the rules of a staircase tableau. Therefore the probability is 0. $\square$

2.3 Boxes on the kth Diagonal

In this section, the distribution of boxes on the kth diagonal, where $k \geq 2$ is fixed, will be computed asymptotically. In other words, the following will be considered

$$\mathbb{P}(\alpha_{j_1}^k, \alpha_{j_2}^k, \ldots, \alpha_{j_r}^k), \;\; 1 \leq j_1 < \cdots < j_r \leq n-k+1, \text{ and}$$

$$\mathbb{P}(x_{j_1}^k, x_{j_2}^k, \ldots, x_{j_r}^k), \;\; 1 \leq j_1 < \cdots < j_r \leq n-k+1$$

where the remaining $n-r-k+1$ boxes on the kth diagonal are not specified (i.e. may contain an α, β, or be empty). For simplicity's sake, the latter probability will be considered as this case will easily reduce to the α case.

To compute $\mathbb{P}(x_{j_1}^k, x_{j_2}^k, \ldots, x_{j_r}^k)$, consider two cases: $m = 1$ and $m > 1$ where

$$m := \min\{l \geq 1 : \;\; j_l \leq j_{l+1} - k\},$$

i.e. all symbols on the kth diagonal before the mth are within k columns of the subsequent symbol, but the mth is not. Notice that $j_2 - j_1 \leq (k-2), \ldots, j_m - j_{m-1} \leq (k-2)$ and thus,

$$j_m \leq (m-1)(k-2) + j_1. \tag{2.13}$$

Now consider arbitrary $S \in \mathcal{S}_n$ such that $x_{j_1}^k, x_{j_2}^k, \ldots, x_{j_r}^k$. By removing the first $j_1 - 1$ columns (see Equation (2.3)) it can be assumed that $j_1 = 1$ although b now depends on n. It is of interest to keep track of certain symbols in the first $\widehat{k} := k + j_m - 1$ columns of S. It is important to note that since $j_m \leq (m-1)(k-2) + 1$ (by (2.13) and the assumption that $j_1 = 1$), $\widehat{k}$ does not depend on n, but rather on k and m which are both fixed. In addition, note that when $m = 1$, $\widehat{k} = k$.

Define a *restricted* symbol to be any β below a symbol (which by the rules of staircase tableaux cannot be an α) or any α to the right of a symbol (which cannot be a β). The symbols of interest are those which are restricted by $x_1^k, \ldots, x_{j_m}^k$. First of all, notice that there are m α's and m β's on the first diagonal which fit this description. In addition, there are many others which are not on the first diagonal.

In order to classify these symbols, let D be the broken line going vertically from β_1^1 to x_1^k then horizontally from x_1^k to $0_{j_2}^{k+1-j_2}$ then vertically from $0_{j_2}^{k+1-j_2}$ to $x_{j_2}^k$ etc., until the vertical segment reaches $x_{j_m}^k$ and then the last horizontal segment goes from $x_{j_m}^k$ to $\alpha_{k+j_m-1}^1$ (see Figure 2.1 for an example). Then the following definition and lemma allow us to keep track of the remaining restricted symbols.

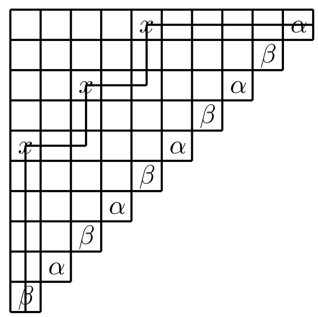


Figure 2.1: A staircase tableau with $m = 3$, $k = 6$, $\widehat{k} = 10$.

Definition 6. *Define a symbol to be D-connected if it is not on the first diagonal and one of the following two conditions hold:*

1. *The symbol lies on D but is not on the kth diagonal.*
2. *There exists a symbol above or to the left that is D-connected or lies on D.*

Lemma 8. *Properties of D-connected symbols:*

1. *Any symbol in the same column as a D-connected α or the same row as a D-connected β is also D-connected (provided it is not on the first diagonal).*
2. *There are at most $\widehat{k} - m - 1$ D-connected symbols.*
3. *Each D-connected symbol can be paired uniquely with an opposite symbol on the first diagonal.*

Proof. Condition (1) follows directly from the definition of D-connected symbols and the rules of staircase tableaux.

Condition (2) follows from the fact that D-connected symbols, by definition, only lie in the region bounded by D and the first diagonal. This region is contained within a subtableau of size $\widehat{k}$ and therefore, D contains at most $2\widehat{k}-1$ symbols. Since there are already m symbols on the kth diagonal and $\widehat{k}$ symbols on the first diagonal, there are at most $\widehat{k}-m-1$ D-connected symbols.

The pairing from Condition (3) can be defined as follows: pair each D-connected β with the first diagonal α it restricts in the same row. Pair each D-connected α with the first diagonal β it restricts in the same column. This pairing is unique since there is only one β in any given row and only one α in any given column. $\square$

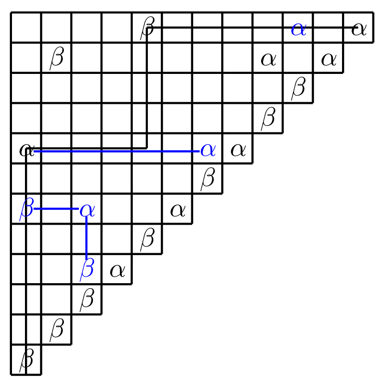


Figure 2.2: A staircase tableau of size 12 with $k = 8$, $m = 2$, and $\widehat{k} = 12$. D is denoted by a black line. There are five D-connected symbols (blue).

In subsequent sections, $\mathbb{P}_{n,\alpha,\beta}(x_1^k, x_{j_2}^k, \ldots, x_{j_r}^k)$ will be calculated by induction. To apply the inductive hypothesis, an arbitrary tableau with r specified symbols on the kth diagonal will be reduced to obtain a new, smaller tableau with $r - m$ specified symbols on the kth diagonal. This can be done by removing some (but not all) of the first $\widehat{k}$ columns and some of the last $\widehat{k}$ rows. In particular, by removing the rows and columns which satisfy the following definition,

Definition 7. *A column (row) is completely restricted if any of the following conditions hold:*

1. *The column (row) contains an α (β) on the kth diagonal.*

2. The column (row) contains a first diagonal α (β) which shares a row (column) with a kth diagonal β (α).

3. There exists a D-connected α (β) in that column (row).

4. There exists a first diagonal α (β) in that column (row) which is paired with a D-connected symbol.

In addition, if the first column contains a kth diagonal α, then the first column and the $(n-k)$th row are completely restricted.

This somewhat lengthy definition was designed to encompass columns and rows in which every symbol in that column or row is restricted by x_1^k, $x_{j_2}^k, \ldots,$ or $x_{j_r}^k$. An additional property is that removing all completely restricted columns and rows preserves the staircase tableau shape (this can be deduced by definition and Lemma 8(3)). Figure 2.3 illustrates these properties.

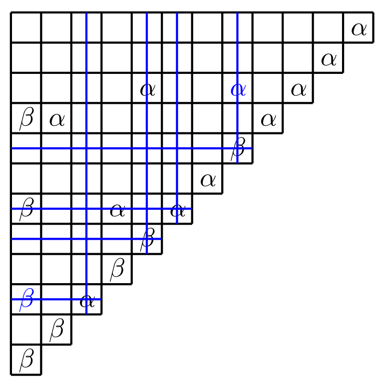


Figure 2.3: A staircase tableau of size 12 with $k = 6$, $m = 2$, $\widehat{k} = 10$ and 2 D-connected symbols (blue). Completely restricted columns and rows are indicated with a blue line. The different ways columns and rows can be completely restricted is illustrated (except the special case when there is an α_1^k).

The information that gets lost when removing all the completely restricted columns is the number of D-connected symbols and their particular arrangement within D. In order to keep track of this information, let $\mathcal{A}_{k,h}$ be the set of all possible arrangements of h D-connected symbols in a tableau of size k (note that D will be clear from the context). For example, $\mathcal{A}_{4,2} = \{\beta_1^2 \cap \beta_1^3, \beta_1^2 \cap \alpha_3^2, \beta_1^3 \cap \alpha_2^3, \beta_2^2 \cap \alpha_2^3, \beta_1^3 \cap \alpha_2^2, \alpha_2^3 \cap \alpha_3^2\}$. Note that $\mathcal{A}_{k,0} = \{\emptyset\}$ for all $k \geq 1$ as there are no arrangements of 0 symbols.

2.3.1 The Asymptotic Distribution of α Boxes

The preceding discussion can now be used to compute the asymptotic distribution of α boxes along the kth diagonal, i.e. $\mathbb{P}_{n,\alpha,\beta}(\alpha^k_{j_1}, \ldots, \alpha^k_{j_r})$. Note that since the kth diagonal will be considered exclusively throughout the remainder of this section, the superscript k will be dropped. The argument will be based on induction and therefore, it will first be discussed how to obtain a new, smaller tableau from an arbitrary tableau $S \in \mathcal{T}_{n,\alpha}$ where $\mathcal{T}_{n,\alpha} := \{S \in \mathcal{S}_{n,\alpha,\beta} : \ \alpha_1, \alpha_{j_2}, \ldots, \alpha_{j_r}\}$. This will be done in two cases.

The case m=1.

Consider an arbitrary $S \in \mathcal{T}_{n,\alpha}$ with h D-connected symbols. Let

$$\mathcal{T}_{n,\alpha,h} := \{S \in \mathcal{S}_{n,\alpha,\beta} : \alpha_{j_2-h-2}, \ldots, \alpha_{j_r-h-2}\},$$

and define $T_S \in \mathcal{T}_{n-h-2,\alpha,h}$ to be the subtableau obtained from S by removing all completely restricted columns and rows. An illustration of this process is given in Figure 2.4.

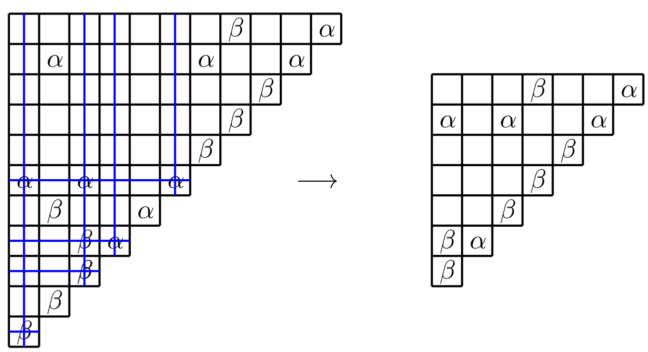


Figure 2.4: An example of the construction of $T_S \in \mathcal{T}_{7,\alpha,2}$ given a tableau $S \in T_{11,\alpha}$ in the case $m = 1$.

This construction implies for any $S \in \mathcal{T}_{n,\alpha}$ with h D-connected symbols

$$wt(S) = \alpha^{h+2}\beta^{h+1}wt(T_S) \tag{2.14}$$

since the weight of the deleted columns and rows is $\alpha^{h+2}\beta^{h+1}$ by Definition 7 and Lemma 8.

Now consider the following lemma,

Lemma 9. *When $m = 1$, there exists a bijection between $T_{n,\alpha}$ and triples (h, a, T) where $0 \leq h \leq k-2$, $a \in \mathcal{A}_{k,h}$ and $T \in \mathcal{T}_{n-h-2,\alpha,h}$.*

Proof. For arbitrary $S \in T_{n,\alpha}$, if h_S is the number of D-connected symbols of S, $a_S \in \mathcal{A}_{k,h}$ is the particular arrangement of those h_S D-connected symbols, and $T_S \in \mathcal{T}_{n-h_S-2,\alpha,h_S}$ is the subtableau of S obtained by removing all completely restricted rows and columns, then

$$F: \ S \mapsto (h_S, a_S, T_S)$$

explicitly defines such a bijection (see Figure 2.5 for an example).

To prove F is injective, consider $S_1, S_2 \in T_{n,\alpha}$ such that $S_1 \neq S_2$. This implies that there exists at least one box in S_1 which is different from S_2, say (i, j) (notice that this box cannot be $(n-k+1, 1)$ as both S_1 and S_2 have an α there). Without loss of generality, assume that box (i, j) in S_1 contains a α and box (i, j) in S_2 is either empty or contains a β. In addition, assume that $h_{S_1} = h_{S_2}$ and $a_{S_1} = a_{S_2}$ since otherwise it is clear that $F(S_1) \neq F(S_2)$.

With these assumptions, the α in S_1 is not D-connected. Therefore by Definition 6, it is not in row $n-k$ and there is no D-connected β in the same row. This is enough (in this context) to conclude that it is not in a completely restricted column or row (see Definition 7). Therefore, this α will appear in the tableau T_{S_1}. Since $a_{S_1} = a_{S_2}$, all the same rows and columns will be deleted in S_2 as in S_1 when obtaining the subtableaux T_{S_1} and T_{S_2}. Thus, the (i, j)th box in S_2 (which is empty or contains a β) will appear in T_{S_2} and in the same position as the α in T_{S_1}. Therefore, $T_{S_1} \neq T_{S_2}$.

Now to prove F is surjective, consider arbitrary triple (h, a, T) where $0 \leq h \leq k-2$, $a \in \mathcal{A}_{k,h}$, and $T \in \mathcal{T}_{n-h-2,\alpha,h}$. The following is a description of how to construct an $S \in \mathcal{T}_{n,\alpha}$ such that $F(S) = (h, a, T)$.

First, construct an (incomplete) staircase tableau of size n such α_1, α_k^1, and β_n^1. In addition, fill in the particular arrangement of D-connected symbols as described by a. Then insert T into boxes which are not on completely restricted columns or rows (T is inserted in order by columns, see Figure 2.6).

The number of completely restricted columns and rows are both $h+2$ in this context. Therefore, the number of boxes which are not on completely restricted columns or rows form the shape of a staircase tableaux of size $n-h-2$, which is exactly the size of T. In addition, inserting T will not violate the rules of staircase tableaux since α_1 and D-connected symbols only restrict D-connected symbols (see Lemma 8(1)) and all of the columns and rows containing D-connected symbols are completely restricted (this can be deduced from Definition 7). Therefore, this construction formulates a staircase tableau, $S \in T_{n,\alpha}$.

$\square$

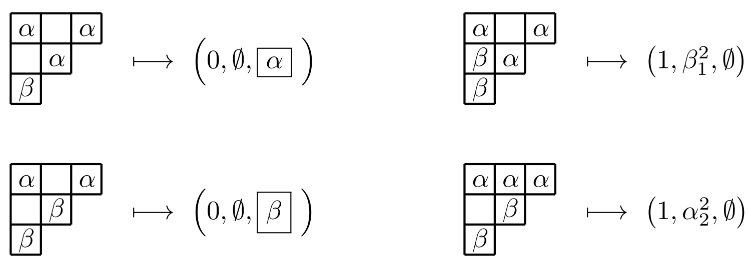


Figure 2.5: The map F from Lemma 9 when $n = 3$.

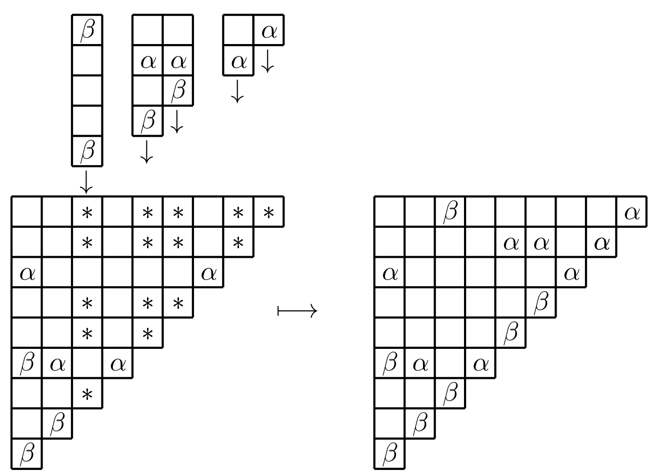


Figure 2.6: An example of the construction of $S \in \mathcal{T}_{9,\alpha}$ given the triple $(2, \beta_1^4 \cap \alpha_2^3, T)$ where $T \in \mathcal{T}_{5,\alpha,2}$ is the tableau being inserted (upper left). Boxes which are not on completely restricted columns or rows are denoted with an "$*$".

Now let

$$C_{k,h} := |\mathcal{A}_{k,h}|,$$

and note that $C_{k,h}$ depends on k and h but not on n (and $C_{k,0} = 1$). In addition, note that (2.14) holds for all S with h D-connected symbols regardless of the particular arrangement of those h D-connected symbols. Therefore by Lemma 9 and Equation (2.14),

$$\sum_{S\in\mathcal{T}_{n,\alpha}} wt(S) = \sum_{h=0}^{k-2} C_{k,h}\alpha^{h+2}\beta^{h+1} \sum_{T\in\mathcal{T}_{n-h-2,\alpha,h}} wt(T).$$

By Definition 5, dividing the total number of weights by

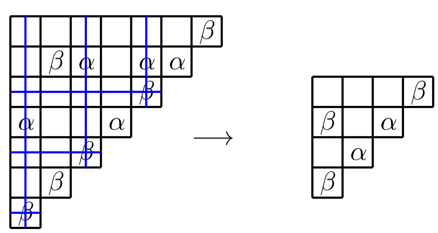
$$Z_n(\alpha,\beta) = (\alpha\beta)^{h+2}(a+b+n-1)_{h+2}Z_{n-h-2}(\alpha,\beta)$$

gives the following

$$\begin{aligned}\mathbb{P}_{n,\alpha,\beta}(\alpha_1,\alpha_{j_2},\ldots,\alpha_{j_r},m=1) &= \\ &\sum_{h=0}^{k-2} C_{k,h}\frac{b}{(n+a+b-1)_{h+2}}\mathbb{P}_{n-h-2,\alpha,\beta}(\alpha_{j_2-h-2},\ldots,\alpha_{j_r-h-2}). \quad (2.15)\end{aligned}$$

The case m>1.

In this case, an upper bound will suffice. Consider arbitrary $S \in \mathcal{T}_{n,\alpha}$ with h D-connected symbols. For this case, let $\mathcal{T}_{n,\alpha,h} := \{S \in \mathcal{S}_{n,\alpha,\beta} :\ \alpha_{j_m-h-m},\ldots,\alpha_{j_r-h-m}\}$ and then define $T_S \in \mathcal{T}_{n-h-m,\alpha,h}$ to be the subtableau obtained from S by removing all completely restricted columns and rows. An illustration of this process in the case $m > 1$ is given in Figure 2.7. This construction implies for



in the case $m = 2$.

Figure 2.7: An example of the construction of $T_S \in \mathcal{T}_{4,\alpha,1}$ given a tableau $S \in T_{7,\alpha}$

any $S \in \mathcal{T}_{n,\alpha}$ with h D-connected symbols

$$wt(S) = \alpha^{h+m}\beta^{h+m}wt(T_S) \tag{2.16}$$

since the weight of the deleted columns and rows is $\alpha^{h+m}\beta^{h+m}$ by Definition 7 and Lemma 8. From this discussion, we have the following lemma whose proof is analogous to the first part of the proof of Lemma 9.

Lemma 10. *When $m > 1$, there exists a injective map between $\mathcal{T}_{n,\alpha}$ and triples (h, a, T) where $0 \leq h \leq \widehat{k} - m - 1$, $a \in \mathcal{A}_{\widehat{k},h}$ and $T \in \mathcal{T}_{n-h-m,\alpha,h}$.*

It is perhaps worthwhile to note that the map from Lemma 10 would be a bijection if the set of tableaux $\mathcal{T}_{n-h-m,\alpha,h}$ was further restricted having some α's on the first diagonal. However, a bijection is not necessary in this case and we prefer to work with the simpler set $\mathcal{T}_{n-h-m,\alpha,h}$.

Therefore, by Lemma 10 and Equation (2.16)

$$\sum_{S \in \mathcal{T}_{n,\alpha}} wt(S) \leq \sum_{h=0}^{\widehat{k}-m-1} C_{\widehat{k},h}\alpha^{h+m}\beta^{h+m} \sum_{T \in \mathcal{T}_{n-h-m,\alpha,h}} wt(T).$$

Notice that $C_{\widehat{k},h}$ does not depend on n since $\widehat{k}$ only depends on k and m. Dividing by $Z_n(\alpha,\beta) = (\alpha\beta)^{h+m}(a+b+n-1)_{h+m}Z_{n-h-m}(\alpha,\beta)$,

$$\begin{aligned}
&\mathbb{P}_{n,\alpha,\beta}(\alpha_1, \alpha_{j_2}, \ldots, \alpha_{j_r}, m > 1) && (2.17)\\
&\leq \sum_{h=0}^{\widehat{k}-m-1} \frac{C_{\widehat{k},h}}{(n+a+b-1)_{h+m}} \mathbb{P}_{n-h-m,\alpha,\beta}(\alpha_{j_{m+1}-h-m}, \ldots, \alpha_{j_r-h-m}) \\
&= O\left(\frac{1}{(n+b)^m}\right) \sum_{h=0}^{\widehat{k}-m-1} \mathbb{P}_{n-h-m,\alpha,\beta}(\alpha_{j_{m+1}-h-m}, \ldots, \alpha_{j_r-h-m}). && (2.18)
\end{aligned}$$

Now, Equation 2.18 can be used to prove the following theorem which gives the asymptotic distribution of α boxes.

Theorem 11. *Let* $1 \leq j_1 < ... < j_r \leq n-k+1$. *If*

$$j_l \leq j_{l+1} - k, \;\; \forall l = 1, 2, ..., r-1$$

then

$$\mathbb{P}_{n,\alpha,\beta}(\alpha_{j_1}, ..., \alpha_{j_r}) = \prod_{l=1}^{r} \frac{j_{r-l+1}}{n^2} + O\left(\frac{1}{n^{r+1}}\right).$$

Otherwise,

$$\mathbb{P}_{n,\alpha,\beta}(\alpha_{j_1}, ..., \alpha_{j_r}) = O\left(\frac{1}{n^r}\right).$$

Proof. In this proof, b will sometimes depend on n, so we will first prove that if $j_l \leq j_{l+1} - k, \;\; \forall\, l = 1, 2, \ldots, r-1$, then

$$\mathbb{P}_{n,\alpha,\beta}(\alpha_{j_1}, ..., \alpha_{j_r}) \;\; = \;\; \prod_{l=1}^{r} \frac{b + j_{r-l+1}}{(n+b)^2} + O\left(\frac{1}{(n+b)^{r+1}}\right).$$

Otherwise,

$$\mathbb{P}_{n,\alpha,\beta}(\alpha_{j_1}, ..., \alpha_{j_r}) = O\left(\frac{1}{(n+b)^r}\right).$$

This statement reduces to the desired claim when b is fixed.

In fact, from this statement, it suffices to prove the theorem for $j_1 = 1$ and $b = O(n)$. For if $j_1 > 1$, by Equation (2.3), we can consider $\mathbb{P}_{n-j_1+1,\alpha,\widehat{\beta}}(\alpha_1, \alpha_{j_2-j_1+1} ..., \alpha_{j_r-j_1+1})$ where $\widehat{b} = b + j_1 - 1$ and the conclusion will follow from the $j_1 = 1$ case.

Therefore, without loss of generality, assume $j_1 = 1$ and $b = O(n)$. Then the proof follows by induction on r. When $r = 1$, apply Equation (2.15)

$$\mathbb{P}_{n,\alpha,\beta}(\alpha_1) \;\; = \;\; \sum_{h=0}^{k-2} C_{k,h} \frac{b}{(n+a+b-1)_{h+2}}.$$

Since $C_{k,0} = 1$ and $C_{k,h}$ does not depend on n for any h and k,

$$\mathbb{P}_{n,\alpha,\beta}(\alpha_1) \;\; = \;\; \frac{b}{(n+a+b-1)_2} + O\left(\frac{1}{(n+b)^2}\right) \sum_{h=1}^{k-2} C_{k,h}$$

$$= \frac{b}{(n+b)^2} + O\left(\frac{1}{(n+b)^2}\right).$$

Assume the statement holds for integers up to $r-1$ and then consider the following two cases.

Case 1. $m = 1$.

Applying Equation (2.15),

$$\begin{aligned}
\mathbb{P}_{n,\alpha,\beta}(\alpha_1, ..., \alpha_{j_r}) &= \sum_{h=0}^{k-2} C_{k,h} \frac{b}{(n+a+b-1)_{h+2}} \mathbb{P}_{n-h-2,\alpha,\beta}(\alpha_{j_2-h-2}, \ldots, \alpha_{j_r-h-2}) \\
&= \frac{b}{(n+b)^2} \mathbb{P}_{n-h-2,\alpha,\beta}(\alpha_{j_2-2}, \ldots, \alpha_{j_r-2}) \\
&\quad + O\left(\frac{1}{(n+b)^2}\right) \sum_{h=1}^{k-2} \mathbb{P}_{n-h-2,\alpha,\beta}(\alpha_{j_2-h-2}, \ldots, \alpha_{j_r-h-2}).
\end{aligned}$$

By the induction hypothesis, if $j_l \leq j_{l+1} - k$ for all $l = 2, \ldots, r-1$,

$$\begin{aligned}
&\mathbb{P}_{n,\alpha,\beta}(\alpha_1, ..., \alpha_{j_r}) \\
&\quad = \frac{b}{(n+b)^2} \left(\prod_{l=1}^{r-1} \frac{b + j_{r-l+1}}{(n+b)^2} + O\left(\frac{1}{(n+b)^r}\right) \right) + O\left(\frac{1}{(n+b)^2}\right) O\left(\frac{1}{(n+b)^{r-1}}\right) \\
&\quad = \prod_{l=1}^{r} \frac{b + j_{r-l+1}}{(n+b)^2} + O\left(\frac{1}{(n+b)^{r+1}}\right).
\end{aligned}$$

On the other hand, if for some $2 \leq l \leq r-1$, $j_l > j_{l+1} - k$ then by the induction hypothesis again,

$$\begin{aligned}
\mathbb{P}_{n,\alpha,\beta}(\alpha_1, ..., \alpha_{j_r}) &= \frac{b}{(n+b)^2} \left(O\left(\frac{1}{(n+b)^{r-1}}\right) \right) + O\left(\frac{1}{(n+b)^2}\right) O\left(\frac{1}{(n+b)^{r-1}}\right) \\
&= O\left(\frac{1}{(n+b)^r}\right).
\end{aligned}$$

Case 2. $m > 1$.

Notice that the cases $j_l = k$, $l \leq m$ are impossible by the rules of staircase tableaux and thus have probability zero.

For all other cases, by Equation (2.18)

$$\mathbb{P}_{n,\alpha,\beta}(\alpha_1,...,\alpha_{j_r}) = O\left(\frac{1}{(n+b)^m}\right)\sum_{h=0}^{\widehat{k}-m-1}\mathbb{P}_{n-h-m,\alpha,\beta}(\alpha_{j_{m+1}-h-m},\ldots,\alpha_{j_r-h-m}).$$

By the induction hypothesis,

$$\begin{aligned}\mathbb{P}_{n,\alpha,\beta}(\alpha_1,...,\alpha_{j_r}) &= O\left(\frac{1}{(n+b)^m}\right)\sum_{h=0}^{\widehat{k}-m-1} O\left(\frac{1}{(n+b)^{r-m}}\right)\\ &= O\left(\frac{1}{(n+b)^r}\right).\end{aligned}$$

Combining Cases 1 and 2 gives the result. $\square$

2.3.2 The Asymptotic Distribution of Non–Empty Boxes

In a similar manner, the asymptotic distribution of the non-empty boxes along the kth diagonal will be computed, i.e. $\mathbb{P}_{n,\alpha,\beta}(x^k_{j_1},\ldots,x^k_{j_r})$. Again, the superscript k will be dropped. Interestingly enough, it turns out that α and β are asymptotically independent which will be shown in this section. The argument will also be based on induction and thus, it will first be discussed how to obtain a new, smaller tableau from an arbitrary tableau $S \in \mathcal{T}_{n,x}$ where $\mathcal{T}_{n,x} := \{S \in \mathcal{S}_n : x_1, x_{j_2}, \ldots, x_{j_r}\}$. This will be done in two cases.

The case m=1.

If the first symbol is an α, then the probability follows easily from the discussion in Section 2.3.1 and the following equation, analogous to (2.15), is obtained.

$$\mathbb{P}_{n,\alpha,\beta}(\alpha_1, x_{j_2},\ldots,x_{j_r}, m=1) = \sum_{h=0}^{k-2} C_{k,h}\frac{b}{(n+a+b-1)_{h+2}}\mathbb{P}_{n-h-2}(x_{j_2-h-2},\ldots,x_{j_r-h-2}). \quad (2.19)$$

When the first symbol is a β, consider arbitrary $S \in \mathcal{T}_{n,x}$ with h D-connected symbols. Let $\beta_j^i(0_l)$ denote the event that there is a β in column j and diagonal i and there are l empty boxes directly above β in the same column. Then if $\mathcal{T}_{n,x,h} := \{T \in \mathcal{S}_n : \beta_1^1(0_{k-h-2}), x_{j_2-h-1}, \ldots, x_{j_r-h-1}\}$, define $T_S \in \mathcal{T}_{n-h-1,x,h}$ be the subtableau obtained from S by removing all completely restricted columns and rows. An illustration of this process in the case $m = 1$ is given in Figure 2.8

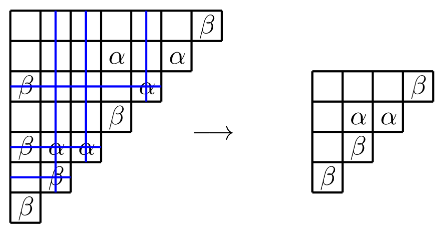


Figure 2.8: An example of the construction of $T_S \in \mathcal{T}_{4,x,2}$ given a tableau $S \in \mathcal{T}_{7,x}$ in the case $m = 1$

This construction implies that for all $S \in \mathcal{T}_{n,x}$ with h D-connected symbols

$$wt(S) = \alpha^{h+1}\beta^{h+1}wt(T_S) \tag{2.20}$$

since the weight of the deleted columns and rows is $\alpha^{h+1}\beta^{h+1}$ by Definition 7 and Lemma 8.

Now consider the following lemma,

Lemma 12. *When $m = 1$, there exists a bijection between $\mathcal{T}_{n,x}$ and triples (h, a, T) where $0 \leq h \leq k-2$, $a \in \mathcal{A}_{k,h}$ and $T \in \mathcal{T}_{n-h-1,x,h}$.*

Proof. The proof is analogous to the proof of Lemma 9 although the construction of an $S \in \mathcal{T}_{n,x}$ given a triple (h, a, T) should be slightly modified. In particular, construct a (incomplete) staircase tableau of size n such that β_1, α_k^1, and β_n^1 and insert the particular arrangement of D-connected symbols as described by a. Then insert T as before except for β_{n-h-1}^1. Figure 2.9 illustrates this process. The proof that this constructs a staircase tableau $S \in \mathcal{T}_{n,x}$ follows similarly to the proof of Lemma 9. $\square$

Now since the weight of a tableau with $\beta_1^1(0_{k-h-2})$ is the same as the weight of a tableau

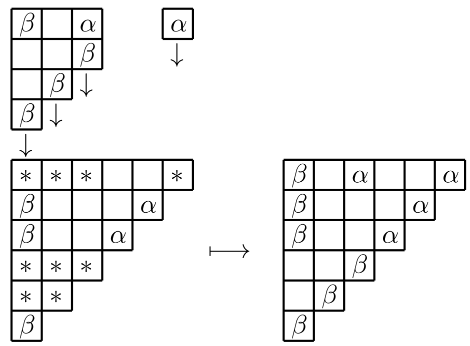


Figure 2.9: An example of the construction of $S \in \mathcal{T}_{6,x}$ given the triple $(1, \beta_1^4, T)$ where $T \in \mathcal{T}_{4,x,1}$ is the tableau being inserted (upper left). Boxes which are not on completely restricted columns and rows are denoted with an "$*$".

with β^1_{k-h-1}, $\mathcal{T}_{n,x,h} = \{T \in \mathcal{S}_n : \beta^1_{k-h-1}, x_{j_2-h-1}, \ldots, x_{j_r-h-1}\}$. Therefore, by Lemma 12 and Equation (2.20)

$$\sum_{S \in \mathcal{T}_{n,x}} wt(S) = \sum_{h=0}^{k-2} C_{k,h} \alpha^{h+1} \beta^{h+1} \sum_{T \in \mathcal{T}_{n-h-1,x,h}} wt(T).$$

Now to simplify the set on the right, consider (for now) tableaux with a fixed number of D-connected symbols and a fixed arrangement of those symbols. In this case, dividing the total number of weights, $\alpha^{h+1}\beta^{h+1} \sum\limits_{T \in \mathcal{T}_{n-h-1,x,h}} wt(T)$, by $Z_n(\alpha,\beta) = (\alpha\beta)^{h+1}(a+b+n-1)_{h+1} Z_{n-h-1}(\alpha,\beta)$ gives the following,

$$\begin{aligned}
\mathbb{P}_{n,\alpha,\beta}(\beta_1, x_{j_2}, \ldots, x_{j_r}) &= \frac{1}{(a+b+n-1)_{h+1}} \mathbb{P}_{n-h-1,\alpha,\beta}(\beta^1_{k-h-1}, x_{j_2-h-1}, \ldots, x_{j_r-h-1}) \\
&= \frac{1}{(a+b+n-1)_{h+1}} \mathbb{P}_{n-h-1,\alpha,\beta}(x_{j_2-h-1}, \ldots, x_{j_r-h-1}) \\
&\quad - \frac{1}{(a+b+n-1)_{h+1}} \mathbb{P}_{n-h-1,\alpha,\beta}(\alpha^1_{k-h-1}, x_{j_2-h-1}, \ldots, x_{j_r-h-1}).
\end{aligned} \tag{2.21}$$

Now, to compute the probability of the event $\{\alpha^1_{k-h-1}, x_{j_2-h-1}, \ldots, x_{j_r-h-1}\}$, delete the first $k-h-2$ columns using (2.3), defining $\widetilde{b} := b + k - h - 2$, $\widetilde{n} := n - k + h + 2$, and $\widetilde{j} := j - k + h + 2$. Then,

$$\mathbb{P}_{n-h-1,\alpha,\beta}(\alpha^1_{k-h-1}, x_{j_2-h-1}, \ldots, x_{j_r-h-1}) \quad = \quad \mathbb{P}_{\widetilde{n}-h-1,\alpha,\widetilde{\beta}}(\alpha^1_1, x_{\widetilde{j}_2-h-1}, \ldots, x_{\widetilde{j}_r-h-1}).$$

Adding the weights of the tableaux such that $\{\alpha^1_1, x_{\widetilde{j}_2-h-1}, \ldots, x_{\widetilde{j}_r-h-1}\}$ and defining $\mathcal{T}_{n,\widetilde{x},h} := \{S \in \mathcal{S}_{n,\alpha,\widetilde{\beta}} : x_{\widetilde{j}_2-h-2}, \ldots, x_{\widetilde{j}_r-h-2}\}$, the following is obtained,

$$\alpha \sum_{T \in \mathcal{T}_{\widetilde{n}-h-2,\widetilde{x}}} wt(T).$$

Dividing by $Z_{\widetilde{n}-h-1}(\alpha, \widetilde{\beta}) = (\alpha\widetilde{\beta})(a + \widetilde{b} + \widetilde{n} - h - 3)Z_{\widetilde{n}-h-2}(\alpha, \widetilde{\beta})$,

$$\begin{aligned}
\mathbb{P}_{n,\alpha,\beta}(\alpha^1_1, x_{\widetilde{j}_2-h-1}, \ldots, x_{\widetilde{j}_r-h-1}) &= \frac{\widetilde{b}}{(a + \widetilde{b} + \widetilde{n} - h - 3)}\mathbb{P}_{\widetilde{n}-h-2,\alpha,\widetilde{\beta}}(x_{\widetilde{j}_2-h-2}, \ldots, x_{\widetilde{j}_r-h-2}) \\
&= \frac{b + k - h - 2}{(a + b + n - h - 3)}\mathbb{P}_{\widetilde{n}-h-2,\alpha,\widetilde{\beta}}(x_{\widetilde{j}_2-h-2}, \ldots, x_{\widetilde{j}_r-h-2}).
\end{aligned}$$

Therefore, returning to Equation (2.21),

$$\begin{aligned}
&\mathbb{P}_{n,\alpha,\beta}(\beta_1, x_{\widetilde{j}_2-h-1}, \ldots, x_{\widetilde{j}_r-h-1}) \\
&\qquad = \frac{1}{(a + b + n - 1)_{h+1}}\mathbb{P}_{n-h-1}(x_{j_2-h-1}, \ldots, x_{j_r-h-1}) \\
&\qquad\quad - \frac{1}{(a + b + n - 1)_{h+1}} \frac{b + k - h - 2}{(a + b + n - h - 3)}\mathbb{P}_{\widetilde{n}-h-2,\alpha,\widetilde{\beta}}(x_{\widetilde{j}_2-h-2}, \ldots, x_{\widetilde{j}_r-h-2}).
\end{aligned}$$

Now if we consider tableaux with any number of D-connected symbols and any arrangement of those D-connected symbols, we obtain our desired calculation

$$\begin{aligned}
&\mathbb{P}_{n,\alpha,\beta}(\beta_1, x_{j_2}, \ldots, x_{j_r}, m = 1) \\
&\qquad = \sum_{h=0}^{k-2} C_{k,h}\left(\frac{1}{(a + b + n - 1)_{h+1}}\mathbb{P}_{n-h-1}\left(x_{j_2-h-1}, \ldots, x_{j_r-h-1}\right)\right. \\
&\qquad\quad \left. - \frac{b + k - h - 2}{(a + b + n - 1)_{h+1}(a + b + n - h - 3)}\mathbb{P}_{\widetilde{n}-h-2,\alpha,\widetilde{\beta}}\left(x_{\widetilde{j}_2-h-2}, \ldots, x_{\widetilde{j}_r-h-2}\right)\right).
\end{aligned} \tag{2.22}$$

The case m>1.

In this case, an upper bound will suffice. Within $x_1, \ldots, x_{j_m}$, say there are l α's and $m-l$ β's for some l. In addition, for this section let $\mathcal{T}_{n,x,h} := \{S \in \mathcal{S}_n : x_{j_{m+1}-h-m}, \ldots, x_{j_r-h-m}\}$. For arbitrary $S \in \mathcal{T}_{n,x}$ with h D-connected symbols let $T_S \in \mathcal{T}_{n-h-m,x,h}$ to be the subtableau obtained from S by removing all completely restricted columns and rows. This construction implies for any $S \in \mathcal{T}_{n,x}$ with h D-connected symbols

$$wt(S) = \alpha^{h+m}\beta^{h+m}wt(T_S) \tag{2.23}$$

since the weight of the deleted columns and rows is $\alpha^{h+m}\beta^{h+m}$ by Definition 7 and Lemma 8. The following lemma stems from this discussion and the proof is analogous to first part of the proof of Lemma 9.

Lemma 13. *When $m > 1$, there exists a injective map between $\mathcal{T}_{n,x}$ and triples (h, a, T) where $0 \le h \le \widehat{k} - m - 1$, $a \in \mathcal{A}_{\widehat{k},h}$ and $T \in \mathcal{T}_{n-h-m,x,h}$.*

Therefore, by Lemma 13

$$\sum_{S \in \mathcal{T}_{n,x}} wt(S) = \sum_{h=0}^{\widehat{k}-m-1} C_{\widehat{k},h}\alpha^{h+m}\beta^{h+m} \sum_{T \in \mathcal{T}_{n-h-m,x,h}} wt(T).$$

Dividing by $Z_n(\alpha,\beta) = (\alpha\beta)^{h+m}(a+b+n-1)_{h+m}Z_{n-h-m}(\alpha,\beta)$ and recalling that $C_{\widehat{k},h}$ does not depend on n,

$$\begin{aligned}
&\mathbb{P}_{n,\alpha,\beta}(x_1, x_{j_2}, \ldots, x_{j_r}, m > 1) \\
&\qquad \le \sum_{h=0}^{\widehat{k}-m-1} \frac{C_{\widehat{k},h}}{(a+b+n-1)_{h+m}} \mathbb{P}_{n-h-m,\alpha,\beta}(x_{j_{m+1}-h-m}, \ldots, x_{j_r-h-m}) \\
&\qquad = O\left(\frac{1}{(n+b)^m}\right) \sum_{h=0}^{\widehat{k}-m-1} \mathbb{P}_{n-h-m,\alpha,\beta}(x_{j_{m+1}-h-m}, \ldots, x_{j_r-h-m}).
\end{aligned} \tag{2.24}$$

We can now give the asymptotic joint distribution of symbols on the kth diagonal. In fact, on the kth diagonal, symbols are asymptotically independent. To show this, we will first compute the probability of the event $\{x_{j_1}, \ldots, x_{j_{r_1+r_2}}\}$, where $0 \leq r_1, r_2$ and $\{x_{j_1}, \ldots, x_{j_{r_1+r_2}}\}$ is a particular sequence of α's and β's. More specifically, we will say that there are r_1 α's and r_2 β's using the notation $\alpha_{j_{\alpha_1}}, \ldots, \alpha_{j_{\alpha_{r_1}}}$, $1 \leq j_{\alpha_1} < \cdots < j_{\alpha_{r_1}} \leq (n-k+1)$ and $\beta_{j_{\beta_1}}, \ldots, \beta_{j_{\beta_{r_1}}}$, $1 \leq j_{\beta_1} < \cdots < j_{\beta_{r_1}} \leq (n-k+1)$, for the ordering of α's and β's separately. As before, we will drop the superscript k.

Theorem 14. *For all* $1 \leq j_1 < \cdots < j_{r_1+r_2} \leq (n-k+1)$. *If,*

$$j_{l+1} - j_l \geq k, \forall l = 1, \ldots, r_1 + r_2 - 1$$

then

$$\begin{aligned}\mathbb{P}_{n,\alpha,\beta}(x_{j_1}, \ldots, x_{j_{r_1+r_2}}) &= \left(\prod_{l_1=1}^{r_1} \mathbb{P}_n(\alpha_{j^a_{r_1-l_1+1}})\right)\left(\prod_{l_2=1}^{r_2} \mathbb{P}_n(\beta_{j^b_{r_2-l_2+1}})\right) + O\left(\frac{1}{n^{r_1+r_2+1}}\right)\\ &= \left(\prod_{l_1=1}^{r_1} \frac{j^a_{r_1-l_1+1}}{n^2}\right)\left(\prod_{l_2=1}^{r_2} \frac{n - j^b_{r_2-l_2+1}}{n^2})\right) + O\left(\frac{1}{n^{r_1+r_2+1}}\right).\end{aligned}$$

Otherwise,

$$\mathbb{P}_{n,\alpha,\beta}(x_{j_1}, \ldots, x_{j_{r_1+r_2}}) = O\left(\frac{1}{n^{r_1+r_2}}\right).$$

Proof. As in Theorem 11, it suffices to prove the theorem for $j_1 = 1$ by allowing b to depend on n. Thus, we will first prove that if $j_{l+1} - j_l \geq k, \forall l = 1, \ldots, r_1 + r_2 - 1$, then

$$\mathbb{P}_n(x_1, \ldots, x_{j_{r_1+r_2}}) = \left(\prod_{l_1=1}^{r_1} \frac{b + j^a_{r_1-l_1+1}}{(n+b)^2}\right)\left(\prod_{l_2=1}^{r_2} \frac{n - j^b_{r_2-l_2+1}}{(n+b)^2})\right) + O\left(\frac{1}{(n+b)^{r_1+r_2+1}}\right)$$

Otherwise,

$$\mathbb{P}_{n,\alpha,\beta}(x_1, \ldots, x_{j_{r_1+r_2}}) = O\left(\frac{1}{(n+b)^{r_1+r_2}}\right).$$

This statement reduces to the desired claim when b is fixed.

The proof will be by induction on $r_1 + r_2$. If $r_1 + r_2 = 1$, then $j_1 = j_{\alpha_1}$ or $j_1 = j_{\beta_1}$ and in either case the conclusion is trivial due to Equation (2.2).

Assume the statement holds for integers up to $r_1 + r_2 - 1$ and consider the following two cases:

Case 3. $m = 1$.

This case will be broken up into two possibilities:

Case 3.1. $j_1 = j_{\alpha_1}$

Applying Equation (2.19) and recalling $b = O(n)$ and $C_{k,h}$ does not depend on n,

$$\begin{aligned}
\mathbb{P}_{n,\alpha,\beta}(\alpha_1, x_{j_2}, \ldots, x_{j_{r_1+r_2}}) &= \sum_{h=0}^{k-2} C_{k,h} \frac{b}{(n+a+b-1)_{h+2}} \mathbb{P}_{n-h-2}(x_{j_2-h-2}, \ldots, x_{j_{r_1+r_2}-h-2}) \\
&= \frac{b}{(n+b)^2} \mathbb{P}_{n-h-2}(x_{j_2-h-2}, \ldots, x_{j_{r_1+r_2}-h-2}) \\
&\quad + O\left(\frac{1}{(n+b)^2}\right) \mathbb{P}_{n-h-2}(x_{j_2-h-2}, \ldots, x_{j_{r_1+r_2}-h-2}).
\end{aligned}$$

By the induction hypothesis, if $j_l \le j_{l+1} - k$ for all $l = 1, \ldots, r_1 + r_2 - 1$ then this is

$$\begin{aligned}
\mathbb{P}_{n,\alpha,\beta}(\alpha_1, x_{j_2}, \ldots, x_{j_{r_1+r_2}}) &= \frac{b}{(n+b)^2} \left(\prod_{l_1=1}^{r_1-1} \frac{b + j^a_{r_1-l_1-1}}{(n+b)^2} \prod_{l_2=1}^{r_2} \frac{n - j^b_{r_2-l_2+1}}{(n+b)^2} + O\left(\frac{1}{(n+b)^{r_1+r_2}}\right)\right) \\
&\quad + O\left(\frac{1}{(n+b)^2}\right) O\left(\frac{1}{(n+b)^{r_1+r_2-1}}\right) \\
&= \prod_{l_1=1}^{r_1} \frac{b + j^a_{r_1-l_1-1}}{(n+b)^2} \prod_{l_2=1}^{r_2} \frac{n - j^b_{r_2-l_2+1}}{(n+b)^2} + O\left(\frac{1}{(n+b)^{r_1+r_2+1}}\right).
\end{aligned}$$

On the other hand, if for some $2 \le l \le r - 1, j_l > j_{l+1} - k$ then by the induction hypothesis again, this is

$$\begin{aligned}
\mathbb{P}_{n,\alpha,\beta}(\alpha_1, x_{j_2}, \ldots, x_{j_{r_1+r_2}}) &= \frac{b}{(n+b)^2} O\left(\frac{1}{(n+b)^{r_1+r_2}}\right) + O\left(\frac{1}{(n+b)^2}\right) O\left(\frac{1}{(n+b)^{r_1+r_2}}\right) \\
&= O\left(\frac{1}{(n+b)^{r_1+r_2+1}}\right)
\end{aligned}$$

Case 3.2. $j_1 = j_{\beta_1}$.

Applying Equation (2.22),

$$
\begin{aligned}
&\mathbb{P}_{n,\alpha,\beta}(\beta_1, x_{j_2}, \ldots, x_{j_{r_1+r_2}}) \\
&\qquad = \sum_{h=0}^{k-2} C_{k,h} \Bigg(\frac{1}{(a+b+n-1)_{h+1}} \mathbb{P}_{n-h-1}\left(x_{j_2-h-1}, \ldots, x_{j_r-h-1}\right) \\
&\qquad - \frac{b+k-h-2}{(a+b+n-1)_{h+1}(a+b+n-h-3)} \mathbb{P}_{\widetilde{n}-h-2,\alpha,\widetilde{\beta}}\left(x_{\widetilde{j}_2-h-2}, \ldots, x_{\widetilde{j}_r-h-2}\right)\Bigg) \\
&\qquad = \frac{1}{n+b} \mathbb{P}_{n-2}(x_{j_2-2}, \ldots, x_{j_r-2}) - \frac{b}{(n+b)^2} \mathbb{P}_{\widetilde{n}-2,\alpha,\widetilde{\beta}}\left(x_{\widetilde{j}_2-2}, \ldots, x_{\widetilde{j}_r-2}\right) \\
&\qquad + O\left(\frac{1}{(n+b)^2}\right) \sum_{h=0}^{k-2} C_{k,h} \mathbb{P}_{n-h-1}(x_{j_2-h-1}, \ldots, x_{j_r-h-1}) \\
&\qquad + O\left(\frac{1}{(n+b)^2}\right) \sum_{h=0}^{k-2} C_{k,h} \mathbb{P}_{\widetilde{n}-h-2,\alpha,\widetilde{\beta}}\left(x_{\widetilde{j}_2-h-2}, \ldots, x_{\widetilde{j}_r-h-2}\right).
\end{aligned}
$$

By the induction hypothesis, if $j_l \leq j_{l+1} - k$ for all $l = 1, \ldots, r_1 + r_2 - 1$ then this is

$$
\begin{aligned}
&\mathbb{P}_{n,\alpha,\beta}(\beta_1, x_{j_2}, \ldots, x_{j_{r_1+r_2}}) \\
&\qquad = \frac{1}{n+b} \left(\prod_{l_1=1}^{r_1} \frac{b + j^a_{r_1-l_1+1}}{(n+b)^2} \prod_{l_2=1}^{r_2-1} \frac{n - j^b_{r_2-l_2+1}}{(n+b)^2} + O\left(\frac{1}{(n+b)^{r_1+r_2+1}}\right)\right) \\
&\qquad - \frac{b}{(n+b)^2} \left(\prod_{l_1=1}^{r_1} \frac{b + j^a_{r_1-l_1+1}}{(n+b)^2} \prod_{l_2=1}^{r_2-1} \frac{n - j^b_{r_2-l_2+1}}{(n+b)^2} + O\left(\frac{1}{(n+b)^{r_1+r_2+1}}\right)\right) \\
&\qquad + O\left(\frac{1}{(n+b)^2}\right) \left(O\left(\frac{1}{(n+b)^{r_1+r_2-1}}\right)\right) \\
&\qquad = \frac{n}{(n+b)^2} \left(\prod_{l_1=1}^{r_1} \frac{b + j^a_{r_1-l_1+1}}{(n+b)^2} \prod_{l_2=1}^{r_2-1} \frac{n - j^b_{r_2-l_2+1}}{(n+b)^2} + O\left(\frac{1}{(n+b)^{r_1+r_2+1}}\right)\right) \\
&\qquad + O\left(\frac{1}{(n+b)^{r_1+r_2+1}}\right) \\
&\qquad = \prod_{l_1=1}^{r_1} \frac{b + j^a_{r_1-l_1+1}}{(n+b)^2} \prod_{l_2=1}^{r_2} \frac{n - j^b_{r_2-l_2+1}}{(n+b)^2} + O\left(\frac{1}{(n+b)^{r_1+r_2+1}}\right).
\end{aligned}
$$

On the other hand, if for some $2 \leq l \leq r-1, j_l > j_{l+1} - k$ then by the induction hypothesis again,

$$
\begin{aligned}
\mathbb{P}_{n,\alpha,\beta}(\beta_1, x_{j_2}, \ldots, x_{j_{r_1+r_2}}) &= \frac{1}{n+b} O\left(\frac{1}{(n+b)^{r_1+r_2-1}}\right) - \frac{b}{(n+b)^2} O\left(\frac{1}{(n+b)^{r_1+r_2-1}}\right) \\
&\quad + O\left(\frac{1}{(n+b)^2}\right) O\left(\frac{1}{(n+b)^{r_1+r_2-1}}\right)
\end{aligned}
$$

$$= \left(\frac{1}{(n+b)^{r_1+r_2}}\right).$$

Case 4. $m > 1$.

By Equation (2.24),

$$\mathbb{P}_{n,\alpha,\beta}(x_1,\ldots,x_{j_{r_1+r_2}}) = O\left(\frac{1}{(n+b)^m}\right)\sum_{h=0}^{\widehat{k}-m}\mathbb{P}_{n-h-m-l}(x_{j_{m+1}-h-m},\ldots,x_{j_{r_1+r_2}-h-m}).$$

By the induction hypothesis,

$$\begin{aligned}\mathbb{P}_{n,\alpha,\beta}(x_1,\ldots,x_{j_{r_1+r_2}}) &= O\left(\frac{1}{(n+b)^m}\right)\sum_{h=0}^{\widehat{k}-m-1} O\left(\frac{1}{(n+b)^{r_1+r_2-m}}\right)\\ &= O\left(\frac{1}{(n+b)^{r_1+r_2}}\right).\end{aligned}$$

□

2.4 The Asymptotic Distribution of Symbols

We can now give the asymptotic distribution of the number of α's on the kth diagonal.

Theorem 15. *Let $A_n^{(k)}$ be the number of α's on the kth diagonal of a random weighted staircase tableau. Then, as $n \to \infty$,*

$$A_n^{(k)} \xrightarrow{d} Pois\left(\frac{1}{2}\right).$$

Proof. By [4 Theorem 22, Chapter 1] and Lemma 1, it suffices to show that for all r,

$$\sum_{1\le j_1<\ldots<j_r\le n-k+1}\mathbb{P}(\alpha_{j_1},\ldots,\alpha_{j_r}) \to \frac{1}{2^r r!}$$

as $n \to \infty$. Write

$$\sum_{1\le j_1<\ldots<j_r\le n-k+1}\mathbb{P}(\alpha_{j_1},\ldots,\alpha_{j_r}) = \sum_J \mathbb{P}(\alpha_{j_1},\ldots,\alpha_{j_r}) + \sum_{J^c}\mathbb{P}(\alpha_{j_1},\ldots,\alpha_{j_r}),$$

where

$$J := \{1 \le j_1 < \cdots < j_r \le n-k+1 :\ \forall\, l = 1, \ldots, r-1;\ j_{l+1} - j_l \ge k\}$$

and

$$J^c := \{1 \le j_1 < \cdots < j_r \le n-k+1 :\ \exists\, l = 1, \ldots, r-1;\ j_{l+1} - j_l < k\}.$$

In addition, recall that

$$J_{r,m} := \{1 \le j_1 < ... < j_r \le m :\ j_k \le j_{k+1} - 2,\ \ \forall k = 1, 2, ..., r-1\}.$$

Then by Theorem 11,

$$\sum_J \mathbb{P}(\alpha_{j_1}, \ldots, \alpha_{j_r}) = \sum_J \left(\prod_{l=1}^{r} \frac{j_{r-l+1}}{n^2} + O\left(\frac{1}{n^{r+1}}\right)\right)$$
$$= \sum_J \prod_{l=1}^{r} \frac{j_l}{n^2} + \binom{n-k+1}{r} \cdot O\left(\frac{1}{n^{r+1}}\right) = \sum_{J_{r,n-k+1}} \prod_{l=1}^{r} \frac{j_l}{n^2} - \sum_{J_{r,n-k+1} \setminus J} \prod_{l=1}^{r} \frac{j_l}{n^2} + O\left(\frac{1}{n}\right).$$

If $(j_1, \ldots, j_r) \in J_{r,n-k+1} \setminus J$ then there exists an l such that $j_{l+1} - j_l < k$ and thus this set has $O\left(\binom{n-k+1}{r-1}\right)$ elements. By this fact and Lemma 2,

$$\sum_J \mathbb{P}(\alpha_{j_1}, \ldots, \alpha_{j_r}) = \frac{1}{2^r r!} + O\left(\binom{n-k+1}{r-1} \cdot \frac{n^r}{n^{2r}}\right) + O\left(\frac{1}{n}\right) = \frac{1}{2^r r!} + O\left(\frac{1}{n}\right).$$

Finally, by Theorem 11,

$$\sum_{J^c} \mathbb{P}(\alpha_{j_1}, \ldots, \alpha_{j_r}) = O\left(\binom{n-k+1}{r-1} \frac{1}{n^r}\right) = O\left(\frac{1}{n}\right).$$

Combining these expressions and taking the limit as $n \to \infty$ completes the proof. □

Corollary 16. *Let $B_n^{(k)}$ to be the number of β's on the kth diagonal of a random weighted staircase tableau. Then, as $n \to \infty$,*

$$B_n^{(k)} \xrightarrow{d} Pois\left(\frac{1}{2}\right).$$

Proof. This follows from the involution discussed in Section 2. □

We can now give the asymptotic distribution of the number of symbols collectively on the kth diagonal. As before, define A_n and B_n to be the number of α's and β's respectively on the kth diagonal, i.e. $A_n := \sum_{j=1}^{n-k+2} I_{\alpha_j}$ and $B_n := \sum_{j=1}^{n-k+2} I_{\beta_j}$. Also, let

$$J = \{1 \leq j_{\alpha_1} < \cdots < j_{\alpha_{r_1}} \leq (n-k+1), 1 \leq j_{\beta_1} < \cdots < j_{\beta_{r_2}} \leq (n-k+1) :$$
$$\forall l = 1, \ldots, r-1; j_{l+1} - j_l \geq k\}$$

and

$$J^c = \{1 \leq j_{\alpha_1} < \cdots < j_{\alpha_{r_1}} \leq (n-k+1), 1 \leq j_{\beta_1} < \cdots < j_{\beta_{r_2}} \leq (n-k+1) :$$
$$\exists l = 1, \ldots, r-1; j_{l+1} - j_l < k\}.$$

Then we have the following,

Theorem 17. *Let $Pois(\lambda)$ be a Poisson random variable with parameter λ. Then, as $n \to \infty$,*

$$(A_n, B_n) \xrightarrow{d} (Z_1, Z_2)\,. \tag{2.25}$$

where $Z_i \in Pois\left(\frac{1}{2}\right)$ are independent random variables.

Proof. By [4 Theorem 23, Chapter 6] and Lemma 1 it suffices to show that for all $r_1, r_2 \geq 0$,

$$\sum_{J \cup J^C} \mathbb{P}_n(x_1, \ldots, x_{j_{r_1+r_2}}) \to \frac{1}{r_1! r_2! \cdot 2^{r_1} 2^{r_2}} \qquad \text{as } n \to \infty. \tag{2.26}$$

First sum over J. By Theorem 14,

$$\begin{aligned} \sum_J \mathbb{P}_n(x_1, \ldots, x_{j_{r_1+r_2}}) &= \sum_J \left(\left(\prod_{l=1}^{r_1} \frac{j_{\alpha_l}}{n^2} \right) \left(\prod_{l=1}^{r_2} \frac{n - j_{\beta_l}}{n^2} \right) + O\left(\frac{1}{n^{r_1+r_2+1}} \right) \right) \\ &= \sum_J \left(\prod_{l=1}^{r_1} \frac{j_{\alpha_l}}{n^2} \right) \left(\prod_{l=1}^{r_2} \frac{n - j_{\beta_l}}{n^2} \right) + \binom{n-k+1}{r_1+r_2} \cdot O\left(\frac{1}{n^{r_1+r_2+1}} \right) \end{aligned}$$

Now for all $(j_1, \ldots, j_{r_1+r_2}) \in J_{r_1,n-k+2} \times J_{r_2,n-k+2} \backslash J$, there exists an l such that $j_{l+1} - j_l < k$ and therefore this set has $O\left(\binom{n-k+1}{r_1+r_2-1}\right)$ elements. Thus, the sum above can written as

$$\sum_{\substack{J_{r_1,n-k+2}\\J_{r_2,n-k+2}}} \left(\prod_{l=1}^{r_1} \frac{j_{\alpha_l}}{n^2}\right)\left(\prod_{l=1}^{r_2} \frac{n-j_{\beta_l}}{n^2}\right) + O\left(\binom{n-k+1}{r_1+r_2-1}\frac{1}{n^{r_1+r_2}}\right)$$

Therefore,

$$\begin{aligned}
&\sum_J \mathbb{P}_n(x_1, \ldots, x_{j_{r_1+r_2}}) \\
&\quad= \left(\sum_{J_{r_1,n-k+2}} \prod_{l=1}^{r_1} \frac{j_{\alpha_l}}{n^2}\right)\left(\sum_{J_{r_2,n-k+2}} \prod_{l=1}^{r_2} \frac{n-j_{\beta_l}}{n^2}\right) + O\left(\frac{1}{n}\right)\\
&\quad= \left(\sum_{J_{r_1,n-k+2}} \prod_{l=1}^{r_1} \frac{j_{\alpha_l}}{n^2}\right)\left(\sum_{k-1\le j_1^b<\cdots<j_{r_2}^b\le n-1} \prod_{l=1}^{r_2} \frac{j_{\beta_l}}{n^2}\right) + O\left(\frac{1}{n}\right)\\
&\quad= \left(\sum_{J_{r_1,n-k+2}} \prod_{l=1}^{r_1} \frac{j_{\alpha_l}}{n^2}\right)\left(\sum_{1\le j_1^b<\cdots<j_{r_2}^b\le n-1} \prod_{l=1}^{r_2} \frac{j_{\beta_l}}{n^2} - \sum_{1\le j_1^b<\cdots<j_{r_2}^b\le k-1} \prod_{l=1}^{r_2} \frac{j_{\beta_l}}{n^2}\right) + O\left(\frac{1}{n}\right).
\end{aligned}$$

Then by Lemma 2,

$$\begin{aligned}
\sum_J \mathbb{P}_n(x_1, \ldots, x_{j_{r_1+r_2}}) &= \left(\frac{1}{n^{2r_1}}\frac{(n-k)_{2r_1}}{2^{r_1}r_1!}\right)\left(\frac{1}{n^{2r_2}}\frac{(n)_{2r_2}}{2^{r_2}r_2!} - \frac{1}{n^{2r_2}}\frac{(k)_{2r_2}}{2^{r_2}r_1!}\right) + O\left(\frac{1}{n}\right)\\
&= \left(\frac{1}{2^{r_1}r_1!}\right)\left(\frac{1}{2^{r_2}r_2!}\right) + O\left(\frac{1}{n}\right)\\
&= \frac{1}{2^{r_1+r_2}r_1!r_2!} + O\left(\frac{1}{n}\right)
\end{aligned}$$

Now for all $(j_1, \ldots, j_{r_1+r_2}) \in J^c$, there exists an l such that $j_{l+1} - j_l < k$ and thus this set has $O\left(\binom{n-k+1}{r_1+r_2-1}\right)$ elements. Therefore, summing over J^c,

$$\sum_{J^C} \mathbb{P}_n(x_{j_1}, \ldots, x_{j_{r_1+r_2}}) = O\left(\binom{n-k+1}{r_1+r_2-1}\frac{1}{n^{r_1+r_2}}\right) = O\left(\frac{1}{n}\right).$$

By (2.26), combining these expressions and letting $n \to \infty$ completes the proof. $\square$

Chapter 3: Tree–Like Tableaux

In this chapter, results on tree–like tableaux will be presented. Prior to this research, the number of occupied corners in tree–like tableaux and the number of occupied corners in symmetric tree–like tableaux were computed [31], and it was conjectured (see Conjectures 4.1 and 4.2 in [31]) that the total number of corners in tree–like tableaux of size n is $n! \times \frac{n+4}{6}$ and the total number of corners in symmetric tree–like tableaux of size $2n+1$ is $2^n \times n! \times \frac{4n+13}{12}$. In this chapter, both conjectures will be proven. The proofs are based on the bijection with permutation tableaux or type–B permutation tableaux (as discussed in Section 1.3) and consequently, results for these tableaux are obtained (see Theorems 21 and 28 below for precise statements). It should be noted that Gao, et. al were able to subsequently prove the conjectures for tree–like tableaux using a different method (see [20 Theorem 4.1] and [20 Theorem 4.3]).

In addition, the limiting distribution of the number of occupied corners in random tree–like tableaux and random symmetric tree–like tableaux will be derived. Laborde Zubieta [31] showed that on average a tree–like tableau has one occupied corner (regardless of the size of the tableau). He also showed that the variance of the number of occupied corners in a random tree–like tableau of size n is $1 - 2/n$ and obtained similar results for the symmetric tree–like tableaux. This suggests that the asymptotic distribution of the number of occupied corners of either type of tableaux is Poisson and this is, indeed, the case.

Lastly, the limiting distribution of the number of diagonal boxes in symmetric tree–like tableaux will be derived. Generating polynomials for the number of diagonal cells in symmetric tree–like tableaux were considered in [1], and the expected number of diagonals in a tableaux of size $2n+1$ was proven to be $3(n+1)/4$ (see [1 Proposition 19]. These results will be extended to prove that the number of diagonal cells is asymptotically normal.

3.1 Preliminaries for Tree–Like Tableaux

Recall from Section 1.1 that corners of a tableau are the cells in which both the right and bottom edges are border edges (i.e. a south step followed by a west step). For convenience, let S_k indicate that the k^{th} step (border edge) is south and W_k indicate that the k^{th} step is west. Thus

$$C_n = \sum_{k=1}^{n-1} I_{S_k, W_{k+1}}, \tag{3.1}$$

where I_A is the indicator random variable of the event A.

The proofs in this chapter will rely on techniques developed in [8] (see also [23]). These two papers used probabilistic language which will be adopted here as well. Thus, instead of talking about the number of corners in tableaux, let $\mathbb{P}_n$ be the uniform probability measure on $\mathcal{X}_n$ (where $\mathcal{X}_n \in \{\mathcal{T}_n, \mathcal{P}_n\}$ will be clear in context) and consider the random variable C_n on the probability space $(\mathcal{X}_n, \mathbb{P}_n)$ where $C_n(T)$ is the number of corners of T. A tableau chosen from $\mathcal{X}_n$ according to the probability measure $\mathbb{P}_n$ is usually referred to as a random tableau of size n and C_n is referred to as the number of corners in a random tableau of size n. Let $\mathbb{E}_n$ denote the expected value with respect to the measure $\mathbb{P}_n$. If $c(\mathcal{X}_n)$ denotes the total number of corners in tableaux in $\mathcal{X}_n$ then the following relation is clear:

$$\mathbb{E}_n C_n = \frac{c(\mathcal{X}_n)}{|\mathcal{X}_n|} \quad \text{or, equivalently,} \quad c(\mathcal{X}_n) = |\mathcal{X}_n|\, \mathbb{E}_n C_n. \tag{3.2}$$

There are several properties of permutation tableaux that were derived in [8] which will be used in this chapter. These properties were obtained as a consequence of a recursive argument that constructed $\mathcal{P}_n$ (denoted by $\mathcal{T}_{n-1}$ in [8] and [23]) by considering all extensions of tableaux of size $n-1$ to tableaux of size n. Specifically, any tableau in $\mathcal{P}_{n-1}$ can be extended to a tableau of size n by adding a row (south step) or adding a new column (west step) and filling its entries with 0 or 1 according to the rules of permutation tableaux. By a simple counting argument, it was shown that there are $2^{U_{n-1}(P)}$ different extensions of a permutation tableau $P \in \mathcal{P}_{n-1}$ to a permutation tableau of size n (see [8] or [23 Section 2] for a detailed explanation but it is clear that only one of these extensions added a south step to P. Using this construction, a relationship between the measures $\mathbb{P}_n$ and $\mathbb{P}_{n-1}$ was derived in [8] and is given by [8 Equation (5)] (see also [23 Section 2, Equation (2.1)]),

$$\mathbb{E}_n(X_{n-1}) = \frac{1}{n}\mathbb{E}_{n-1}(2^{U_{n-1}}X_{n-1}) \tag{3.3}$$

where X_{n-1} is any random variable defined on $\mathbb{P}_{n-1}$. Let $\mathcal{F}_{n-1}$ denote the σ-subalgebra on $\mathcal{P}_n$ obtained by grouping together all tableaux of size n obtained from the same tableau of size $n-1$. The conditional distribution of U_n given $\mathcal{F}_{n-1}$ was determined to be the following,

$$\mathcal{L}(U_n|\mathcal{F}_{n-1}) = 1 + \mathrm{Bin}(U_{n-1}), \tag{3.4}$$

where $\mathrm{Bin}(m)$ denotes a binomial random variable with parameters m and $1/2$ (see [8]).

3.2 Corners in Tree–Like Tableaux

The main result of this section is the proof of the first conjecture of Laborde Zubieta.

Theorem 18. *(see [31 Conjecture 4.1]) For $n \geq 2$ we have*

$$c(\mathcal{T}_n) = n! \times \frac{n+4}{6}.$$

The proof will use a bijection between tree–like tableaux and permutation tableaux described in [1]. According to Proposition 1.3 of [1], there exists a bijection between permutation tableaux and tree–like tableaux which transforms a tree–like tableau of shape F to a permutation tableau of shape F' which is obtained from F by removing the SW–most edge from F and the cells of the left–most column (see Figure 3.1).

The number of corners in F is the same as the number of corners in F' if the last edge of F' is horizontal and it is one more than the number of corners in F' if the last edge of F' is vertical. Furthermore, as is clear from a recursive construction described in [8 Section 2], any permutation tableau of size n whose last edge is vertical is obtained as the unique extension of a permutation

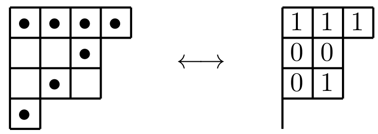


Figure 3.1: An example of the bijection between permutation tableaux and tree–like tableaux of size 7.

tableau of size $n-1$. Therefore, there are $(n-1)!$ such tableaux and we have a simple relation

$$c(\mathcal{T}_n) = c(\mathcal{P}_n) + |\{P \in \mathcal{P}_n : \ M_n(P) = S\}| = c(\mathcal{P}_n) + (n-1)!. \tag{3.5}$$

Thus, it suffices to determine the number of corners in the permutation tableaux of size n or the expected number of corners since $|\mathcal{P}_n| = n!$ and therefore equation (3.2) becomes

$$c(\mathcal{P}_n) = n!\,\mathbb{E}_n C_n. \tag{3.6}$$

Now consider the following preliminary theorem.

Theorem 19. *For permutation tableaux of size n, the probability of having a corner with border edges k and $k+1$ is given by*

$$\mathbb{P}_n\left(I_{S_k,W_{k+1}}\right) = \frac{n-k+1}{n} - \frac{(n-k)^2}{n(n-1)}.$$

Proof. The theorem can be proven using the techniques developed in [8]. Specifically, if $k+1 \leq n-1$ then $I_{S_k,W_{k+1}}$ is a random variable on $\mathcal{P}_{n-1}$. Therefore, by (3.3)

$$\begin{aligned}
\mathbb{P}_n\left(I_{S_k,W_{k+1}}\right) &= \mathbb{E}_n\left(I_{S_k,W_{k+1}}\right) = \frac{1}{n}\mathbb{E}_{n-1}\left(2^{U_{n-1}} I_{S_k,W_{k+1}}\right) \\
&= \frac{1}{n}\mathbb{E}_{n-1}\mathbb{E}\left(2^{U_{n-1}} I_{S_k,W_{k+1}} | \mathcal{F}_{n-2}\right),
\end{aligned}$$

where $\mathcal{F}_{n-2}$ is a σ–subalgebra on $\mathcal{P}_{n-1}$ obtained by grouping into one set all tableaux in $\mathcal{P}_{n-1}$ that are obtained by extending the same tableau in $\mathcal{P}_{n-2}$ (see Section 3.2). Now, if $k+1 \leq n-2$ then $I_{S_k,W_{k+1}}$ is measurable with respect to the σ-algebra $\mathcal{F}_{n-2}$. Thus by the properties of conditional expectation the above is:

$$\mathbb{E}_n\left(I_{S_k,W_{k+1}}\right) \quad = \quad \frac{1}{n}\mathbb{E}_{n-1}I_{S_k,W_{k+1}}\mathbb{E}\left(2^{U_{n-1}}|\mathcal{F}_{n-2}\right).$$

By (3.4) and the fact that $\mathbb{E}a^{\mathrm{Bin}(m)} = \left(\frac{a+1}{2}\right)^m$, the following can be obtained by the same computation as in [23] (see (2.2) and (2.3) there),

$$\begin{aligned}
\frac{1}{n}\mathbb{E}_{n-1}I_{S_k,W_{k+1}}\mathbb{E}\left(2^{U_{n-1}}|\mathcal{F}_{n-2}\right) &= \frac{1}{n}\mathbb{E}_{n-1}I_{S_k,W_{k+1}}\mathbb{E}\left(2^{1+\mathrm{Bin}(U_{n-2})}|\mathcal{F}_{n-2}\right) \\
&= \frac{2}{n}\mathbb{E}_{n-1}I_{S_k,W_{k+1}}\left(\frac{3}{2}\right)^{U_{n-2}} \\
&= \frac{2}{n(n-1)}\mathbb{E}_{n-2}I_{S_k,W_{k+1}}3^{U_n-2} \qquad (3.7)
\end{aligned}$$

where the last step follows from (3.3).

Iterating $(n-1)-(k+1)$ times,

$$\frac{2\cdot 3\cdot\cdots\cdot(n-k-1)}{n(n-1)\cdot\cdots\cdot(k+2)}\mathbb{E}_{k+1}I_{S_k,W_{k+1}}\left(n-k\right)^{U_{k+1}}. \qquad (3.8)$$

Now consider

$$\mathbb{E}_{k+1}I_{S_k,W_{k+1}}(n-k)^{U_{k+1}} \qquad (3.9)$$

for $1 \leq k \leq n-1$ (note that $k+1=n$ gives $\mathbb{E}_nI_{S_{n-1},W_n}$ which is exactly the summand omitted earlier by the restriction $k+1 \leq n-1$). This can be computed as follows. First, by the tower property of the conditional expectation and the fact that S_k is $\mathcal{F}_k$–measurable,

$$\mathbb{E}_{k+1}I_{S_k,W_{k+1}}(n-k)^{U_{k+1}} = \mathbb{E}_{k+1}I_{S_k}\mathbb{E}(I_{W_{k+1}}(n-k)^{U_{k+1}}|\mathcal{F}_k).$$

And now

$$\mathbb{E}(I_{W_{k+1}}(n-k)^{U_{k+1}}|\mathcal{F}_k) = \mathbb{E}((n-k)^{U_{k+1}}|\mathcal{F}_k) - \mathbb{E}(I_{S_{k+1}}(n-k)^{U_{k+1}}|\mathcal{F}_k)$$

because the two indicators are complementary. By the same computation as above (see (3.7)), the first conditional expectation on the right–hand side is

$$(n-k)\mathbb{E}\left((n-k)^{U_{k+1}}|\mathcal{F}_k\right) = (n-k)\left(\frac{n-k+1}{2}\right)^{U_k}. \tag{3.10}$$

To compute the second conditional expectation, note that on the set S_{k+1}, $U_{k+1} = 1 + U_k$ so that

$$\begin{aligned}
\mathbb{E}(I_{S_{k+1}}(n-k)^{U_{k+1}}|\mathcal{F}_k) &= (n-k)^{1+U_k}\mathbb{E}(I_{S_{k+1}}|\mathcal{F}_k) \\
&= (n-k)^{1+U_k}\mathbb{P}(I_{S_{k+1}}|\mathcal{F}_k) \\
&= (n-k)^{1+U_k}\frac{1}{2^{U_k}}
\end{aligned}$$

where the last equation follows from the fact that for every tableau $P \in \mathcal{P}_k$ only one of its $2^{U_k(P)}$ extensions to a tableau in $\mathcal{P}_{k+1}$ has S_{k+1} (see Section 3.2 and also [8; 23] for more details). Combining with (3.10) yields

$$\mathbb{E}(I_{W_{k+1}}(n-k)^{U_{k+1}}|\mathcal{F}_k) = (n-k)\left(\left(\frac{n-k+1}{2}\right)^{U_k} - \left(\frac{n-k}{2}\right)^{U_k}\right)$$

and thus (3.9) equals

$$(n-k)\mathbb{E}_{k+1}\left(I_{S_k}\left(\left(\frac{n-k+1}{2}\right)^{U_k} - \left(\frac{n-k}{2}\right)^{U_k}\right)\right).$$

The expression inside the expectation is a random variable on $\mathcal{P}_k$, therefore, (3.3) can be used to reduce the size by one and show that the expression above is

$$\frac{n-k}{k+1}\mathbb{E}_k I_{S_k}\left((n-k+1)^{U_k} - (n-k)^{U_k}\right).$$

Furthermore, on the set S_k, $U_k = U_{k-1} + 1$ so that the above is

$$\frac{n-k}{k+1}\mathbb{E}_k\left(\left((n-k+1)^{1+U_{k-1}} - (n-k)^{1+U_{k-1}}\right)\mathbb{E}(I_{S_k}|\mathcal{F}_{k-1})\right),$$

which, by the same argument as above, equals

$$\frac{n-k}{k+1}\mathbb{E}_k\left(\left((n-k+1)^{1+U_{k-1}} - (n-k)^{1+U_{k-1}}\right)\frac{1}{2^{U_{k-1}}}\right).$$

After reducing the size one more time, the above is

$$\frac{n-k}{(k+1)k}\left(\mathbb{E}_{k-1}(n-k+1)^{1+U_{k-1}} - \mathbb{E}_{k-1}(n-k)^{1+U_{k-1}}\right). \tag{3.11}$$

As computed in [23 Equation (2.4)] for a positive integer m the generating function of U_m is given by

$$\mathbb{E}_m z^{U_m} = \frac{\Gamma(z+m)}{\Gamma(z)m!}.$$

(There is an obvious omission in (2.4) there; the $z+n$ in the third expression should be $z+n-1$.) Using this with $m = k-1$ and $z = n-k+1$ and then with $z = n-k$,

$$\mathbb{E}_{k-1}\left((n-k+1)^{1+U_{k-1}}\right) = (n-k+1)\frac{(n-1)!}{(n-k)!(k-1)!} \tag{3.12}$$

and

$$\mathbb{E}_{k-1}\left((n-k)^{1+U_{k-1}}\right) = (n-k)\frac{(n-2)!}{(n-k-1)!(k-1)!}. \tag{3.13}$$

Combining Equations (3.8), (3.11), (3.12), and (3.13),

$$\begin{aligned}\mathbb{E}_n\left(I_{S_k,W_{k+1}}\right) &= \frac{(n-k-1)!(k+1)!}{n!}\cdot\frac{n-k}{k(k+1)}\left(\frac{(n-k+1)(n-1)!}{(k-1)!(n-k)!} - \frac{(n-k)(n-2)!}{(k-1)!(n-k-1)!}\right)\\ &= \frac{n-k+1}{n} - \frac{(n-k)^2}{n(n-1)},\end{aligned}$$

and the conclusion follows. $\square$

The relationship between permutation tableaux and tree–like tableaux given by (3.5) implies the following corollary to Theorem 3.7.

Corollary 20. *For tree–like tableaux of size n, $n \geq 2$, the probability of having a corner with border edges k and $k+1$ is given by*

$$\mathbb{P}_n\left(I_{S_k,W_{k+1}}\right) = \begin{cases} \frac{n-k+1}{n} - \frac{(n-k)^2}{n(n-1)} & k = 1,\dots,n-1; \\ \frac{1}{n} & k = n. \end{cases}$$

Finally, the following result, when combined with (3.5) and (3.6), completes the proof of Theorem 18.

Theorem 21. *For permutation tableaux of size n,*

$$\mathbb{E}_n C_n = \frac{n+4}{6} - \frac{1}{n}.$$

Proof. In view of (3.1), consider

$$\mathbb{E}_n\left(\sum_{k=1}^{n-1} I_{S_k,W_{k+1}}\right) = \sum_{k=1}^{n-1} \mathbb{E}_n\left(I_{S_k,W_{k+1}}\right).$$

Then the result is obtained by summing the expression from Theorem 19 from $k = 1$ to $n - 1$

$$\begin{aligned} \mathbb{E}_n C_n &= \sum_{k=1}^{n-1} \frac{n-k+1}{n} - \sum_{k=1}^{n-1} \frac{(n-k)^2}{n(n-1)} = \sum_{j=2}^{n} \frac{j}{n} - \sum_{j=1}^{n-1} \frac{j^2}{n(n-1)} \\ &= \frac{n(n+1)}{2n} - \frac{1}{n} - \frac{(n-1)n(2n-1)}{6n(n-1)} = \frac{n+4}{6} - \frac{1}{n} \end{aligned}$$

as desired. ∎

To conclude this section, note that Theorem 18 could also be obtained by summing the expression from Corollary 20 from $k = 1$ to n.

3.3 Preliminaries for the Symmetric Case

The rest of this chapter is devoted to proving the second conjecture of Laborde Zubieta on symmetric tree–like tableaux ([31 Conjecture 4.2]). Analogous to Section 3.2, a bijection will be used between symmetric tree–like tableaux and type–B permutation tableaux to relate the corners of $\mathcal{T}_{2n+1}^{sym}$ to the corners of $\mathcal{B}_n$. In Section 2.2 of [1], it was mentioned that there exists such a bijection; however, no details were given. Thus, the following is a description of one such bijection (see Figure 3.2).

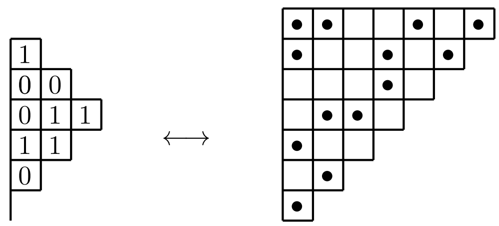


Figure 3.2: An example of the bijection F as defined in Lemma 22 between type–B permutation tableaux of size 6 and symmetric tree–like tableaux of size 13.

Lemma 22. *Consider* $F : \mathcal{T}_{2n+1}^{sym} \to \mathcal{B}_n$ *defined by the following rules,*

1. *Replace the topmost point in each column with* 1_T*'s.*
2. *Replace the leftmost points in each row with* 0_R*'s*
3. *Fill in the remaining cells according to the rules of type–B permutation tableaux.*
4. *Remove the cells above the diagonal.*
5. *Remove the first column.*

and $F^{-1} : \mathcal{B}_n \to \mathcal{T}_{2n+1}^{sym}$ *defined by:*

1. *Add a column and point all cells except those in a restricted row.*
2. *Replace all* 0_R*'s with points unless that* 0_R *is in the same row as a diagonal* 0.
3. *Replace all non-diagonal* 1_T*'s with points.*
4. *Delete the remaining numbers, add a pointed box in the upper–left–hand corner (the root point), and then add the boxes necessary to make the tableau symmetric.*

Then F is a bijection between $\mathcal{T}^{sym}_{2n+1}$ and $\mathcal{B}_n$.

Proof. First, we show that F and F^{-1} are well–defined. For arbitrary $T \in \mathcal{T}^{sym}_{2n+1}$, each column in T contains a point and thus, there is always a topmost point. Therefore, each column in $F(T)$ contains a 1_T and Definition 4(1) is satisfied. The condition (2) in Definition 4 is satisfied since the only zeros in $F(T)$ are from the leftmost pointed cells (therefore, there is no topmost one to the left) or from rule (3) which would not violate this condition. Condition (3) in Definition 4 is satisfied similarly.

For arbitrary $B \in \mathcal{B}_n$, $F^{-1}(B)$ satisfies Definition 2(1) by rule (4). Because of symmetry, $F^{-1}(B)$ satisfies Definition 2(2) if every row contains a point. All restricted rows (excluding rows with a diagonal 0) will get mapped to a pointed row since the 0_R gets pointed by rule (2). All unrestricted rows will get pointed in the new column by rule (1). Now consider the rows which contain a diagonal 0. There must be a 1_T in the column below the diagonal 0 (Definition 4(1)). This cell will get pointed after applying F^{-1}. Therefore, there will be a pointed cell to the right of the diagonal in $F^{-1}(B)$ which is in the same row as the diagonal 0 after applying rule (4). Therefore, all diagonal zero rows also get mapped to pointed rows. Finally, condition Definition 2(3) is met since points added in the new column clearly have no points to the left of them. In addition, the points which come from a 0_R have no points to the left of them and the points which come from a 1_T have no points above them.

Now to prove F is one-to-one, consider arbitrary $B_1, B_2 \in \mathcal{B}_n$ such that $B_1 \neq B_2$. If the Ferrers diagrams of B_1 and B_2 are different, then it is obvious that $F^{-1}(B_1) \neq F^{-1}(B_2)$ since these are tree-like tableaux of different shapes. If the Ferrers diagrams are the same, then there must be at least one cell which is labeled differently. Consider the highest, rightmost such cell, say (i, j). W.L.O.G. assume that $B_1(i, j) = 0$ and $B_2(i, j) = 1$.

Consider two cases.

Case 5. $B_1(i, j) = 0_R$.

In this case, there exists a cell above (i, j) that is filled with a 1 in both B_1 and B_2. By rule (2), $F^{-1}(B_1(i, j))$ is pointed but $F^{-1}(B_2(i, j))$ is not since it is not the highest one in its column (note that this 0_R can't be on a diagonal 0 row since we have picked the highest, rightmost point that is different). Therefore, $F^{-1}(B_1) \neq F^{-1}(B_2)$.

Case 6. $B_1(i,j) \neq 0_R$.

In this case, all cells above (i,j) are filled with 0's in both B_1 and B_2. If such cells exist, by rule (3) $F^{-1}(B_2(i,j))$ is pointed but $F^{-1}(B_1(i,j))$ is not since it is not a restricted zero. If there are no cells above (i,j), then $B_1(i,j) = 0$ is a diagonal 0 and thus, none of the cells in this row get pointed.

But since $B_2(i,j) = 1$, this row is either unrestricted and the added cell (from the added column) gets pointed (rule (1)) or it is a restricted row (and does not contain a diagonal 0) and the 0_R gets pointed (rule (2)). Therefore, $F^{-1}(B_1) \neq F^{-1}(B_2)$. □

The bijection described in Lemma 22 transforms a tree–like tableau of shape F to a permutation tableau of shape F' that is obtained from F by removing all the cells on and above the diagonal of F, removing the SW–most edge from F, and removing the cells of the left–most column of F (see Figure 3.2). The number of corners in F is the same as the number of corners in F' unless the last edge of F' is vertical or the first edge of F' is vertical. In the former case, F has two additional corners. In the latter case, F has one additional corner. This leads to the following relationship,

$$c(\mathcal{T}^{sym}_{2n+1}) = 2c(\mathcal{B}_n) + 2|\{B \in \mathcal{B}_n : \ M_n(B) = S\}| + |\{B \in \mathcal{B}_n : \ M_1(B) = W\}|. \tag{3.14}$$

3.4 An Extension Procedure for Type–B Permutation Tableaux

Now to carry out calculations on type–B permutation tableaux, an extension procedure for type–B permutation tableaux will be constructed which mimics the construction given in [8 Section 2] for permutation tableaux (as described in Section 3.2). First of all, fix any $B \in \mathcal{B}_{n-1}$ and let $U_{n-1} = U_{n-1}(B)$ denote the number of unrestricted rows in B. Note that the size of B can be extended to n by inserting a new row or a new column whose fillings depend on U_{n-1}.

The only way to insert a new row is by adding a south step to the shape. The ways to insert a new column depend on the filling of that column. Any restricted row forces a 0 in the new cell in that row. The remaining $U_{n-1}+1$ cells (the one additional cell is the diagonal cell on the top row) to be filled with either a 1 or 0 so that there is at least one 1. Thus, there are $2^{U_{n-1}+1}-1$ possible fillings of a new column and $2^{U_{n-1}+1}$ different extensions of our tableau to a type–B tableau of size n. Let U_n be the number of unrestricted rows in the extended tableaux, $U_n = 1, \ldots, U_{n-1}+1$. If a row is inserted, then $U_n = U_{n-1}+1$. Since the row is inserted in precisely one of the possible $2^{U_{n-1}+1}$ cases, the (conditional) probability that $U_n = U_{n-1}+1$ is

$$\mathbb{P}(U_n = U_{n-1}+1|\mathcal{F}_{n-1}) = \mathbb{P}(S_n|\mathcal{F}_{n-1}) = \frac{1}{2^{U_{n-1}+1}}. \tag{3.15}$$

(Here, analogously to permutation tableaux (see the proof of Theorem 21 above or [23 Section 2]) $\mathcal{F}_{n-1}$ is a σ–subalgebra on $\mathcal{B}_n$ obtained by grouping together all tableaux in $\mathcal{B}_n$ that are obtained as the extension of the same tableau from $\mathcal{B}_{n-1}$.)

If a column is inserted, the number of unrestricted rows depends on two cases. First, if a 1 is inserted in the new diagonal cell, then any 0 below it in an unrestricted row becomes restricted. Thus, for the extension to have k unrestricted rows, there must be $k-1$ 1's placed below the diagonal cell and there are $\binom{U_{n-1}}{k-1}$ ways do so. If a 0 is inserted in the new diagonal cell, then this reduces to adding a column to a permutation tableaux with U_{n-1} unrestricted rows. The number of ways to do so was already found in [8] and is $\binom{U_{n-1}}{k-1}$. Thus,

$$\mathbb{P}(U_n = k|\mathcal{F}_{n-1}) = \frac{1}{2^{U_{n-1}+1}}\left(\binom{U_{n-1}}{k-1}+\binom{U_{n-1}}{k-1}\right) = \frac{1}{2^{U_{n-1}}}\binom{U_{n-1}}{k-1},$$

for $k = 1, \ldots, U_{n-1}$. This agrees with (3.15) when $k = U_{n+1}$. Thus,

$$\mathcal{L}(U_n|\mathcal{F}_{n-1}) = 1 + \mathrm{Bin}(U_{n-1}),$$

where the left–hand side means the conditional distribution of U_n given U_{n-1} and $\mathrm{Bin}(m)$ denotes a binomial random variable with parameters m and $1/2$. Note that this is the same relationship as for permutation tableaux (see [23 Equation (2.2)] or [8 Equation 4]).

As in the case of permutation tableaux, the uniform measure $\mathbb{P}_n$ on $\mathcal{B}_n$ induces a measure (still denoted by $\mathbb{P}_n$) on $\mathcal{B}_{n-1}$ via a mapping $\mathcal{B}_n \to \mathcal{B}_{n-1}$ that assigns to any $B' \in \mathcal{B}_n$ the unique tableau of size $n-1$ whose extension is B'. These two measures on $\mathcal{B}_{n-1}$ are not identical, but the relationship between them can be easily calculated (see [8 Section 2] or [23 Section 2] for more details and calculations for permutation tableaux). Namely,

$$\mathbb{P}_n(B) \quad = \quad 2^{U_{n-1}(B)+1}\frac{|\mathcal{B}_{n-1}|}{|\mathcal{B}_n|}\mathbb{P}_{n-1}(B), \quad B \in \mathcal{B}_{n-1}.$$

This relationship implies that for any random variable X on $\mathcal{B}_{n-1}$,

$$\mathbb{E}_n X \quad = \quad \frac{2|\mathcal{B}_{n-1}|}{|\mathcal{B}_n|}\mathbb{E}_{n-1}(2^{U_{n-1}(\mathcal{B}_{n-1})}X). \tag{3.16}$$

This allows for a direct proof of the following well known fact which was previously known via bijections (see for example, [9 Proposition 4.1]).

Proposition 23. *For all $n \geq 0$, $|\mathcal{B}_n| = 2^n n!$.*

Proof. By considering all the extensions of a type–B permutation tableaux of size $n-1$, we have the following relationship,

$$|\mathcal{B}_n| = \sum_{B \in \mathcal{B}_{n-1}} 2^{U_{n-1}(B)+1}.$$

Thus,

$$\begin{aligned}
|\mathcal{B}_n| &= |\mathcal{B}_{n-1}|\mathbb{E}_{n-1}\left(2^{U_{n-1}+1}\right) \\
&= 2|\mathcal{B}_{n-1}|\mathbb{E}_{n-1}\mathbb{E}\left(2^{1+\mathrm{Bin}(U_{n-2})}|U_{n-2}\right) \\
&= 2 \cdot 2|\mathcal{B}_{n-1}|\mathbb{E}_{n-1}\left(\frac{3}{2}\right)^{U_{n-2}} \\
&= 2 \cdot 2|\mathcal{B}_{n-1}|\frac{2|\mathcal{B}_{n-2}|}{|\mathcal{B}_{n-1}|}\mathbb{E}_{n-2}\left(2^{U_{n-2}}\left(\frac{3}{2}\right)^{U_{n-2}}\right) \\
&= 2^2 \cdot 2!\,|\mathcal{B}_{n-2}|\mathbb{E}_{n-2}3^{U_{n-2}}.
\end{aligned}$$

Iterating n times,

$$\begin{aligned} |\mathcal{B}_n| &= 2^3 \cdot 3!\,|\mathcal{B}_{n-3}|\mathbb{E}_{n-3}4^{U_{n-3}} = 2^{n-1}(n-1)!|\mathcal{B}_1|\mathbb{E}_1 n^{U_1} \\ &= 2^n n!, \end{aligned}$$

where the final equality holds because $|\mathcal{B}_1| = 2$ and $U_1 \equiv 1$. □

Given Proposition 23 (3.16) reads

$$\mathbb{E}_n X = \frac{1}{n}\mathbb{E}_{n-1}(2^{U_{n-1}(B_{n-1})}X). \tag{3.17}$$

This is exactly the same expression as [8 Equation (7)] which means that the relationship between $\mathbb{E}_n$ and $\mathbb{E}_{n-1}$ is the same regardless of whether we are considering $\mathcal{P}_n$ or $\mathcal{B}_n$. Thus, any computation for B–type tableaux based on (3.17) will lead to the same expression as the analogous computation for permutation tableaux based on [8 Equation (7)].

3.5 Corners in Symmetric Tree–Like Tableaux

The main result of this section is the proof of the second conjecture of Laborde Zubieta.

Theorem 24. *(see [31 Conjecture 4.2]) For $n \geq 2$ we have*

$$c(\mathcal{T}^{sym}_{2n+1}) = 2^n \times n! \times \frac{4n+13}{12}.$$

Analogous to the general case, the proof will use a relationship between symmetric tree–like tableaux and type–B permutation tableaux (namely (3.19)) which was determined by a bijection between them. Due to the extension procedure discussed in Section 3.4, the relation (3.19) can be further simplified.

Lemma 25. *The number of corners in symmetric tree–like tableaux is given by,*

$$c(\mathcal{T}^{sym}_{2n+1}) = 2c(\mathcal{B}_n) + 2^n(n-1)! + 2^{n-1}n!. \tag{3.18}$$

Proof. Recall the relation (3.19),

$$c(\mathcal{T}^{sym}_{2n+1}) = 2c(\mathcal{B}_n) + 2|\{B \in \mathcal{B}_n :\ S_n\}| + |\{B \in \mathcal{B}_n :\ W_1\}|. \tag{3.19}$$

By the extension procedure described in Section 3.5, it is clear that

$$|\{B \in \mathcal{B}_n :\ S_n\}| = |\mathcal{B}_{n-1}| = 2^{n-1}(n-1)!. \tag{3.20}$$

In addition,

$$|\{B \in \mathcal{B}_n :\ W_1\}| = 2^n n!\, \mathbb{E}_n(I_{W_1}).$$

Furthermore, by the same argument as in the proof of Proposition 23:

$$\begin{aligned}
\mathbb{E}_n(I_{W_1}) &= \frac{1}{n}\mathbb{E}_{n-1}(2^{U_{n-1}} I_{W_1}) = \frac{1}{n}\mathbb{E}_{n-1}\left(I_{W_1}\mathbb{E}(2^{U_{n-1}}|U_{n-2})\right) \\
&= \frac{2}{n}\mathbb{E}_{n-2}\left(I_{W_1}3^{U_{n-2}}\right) = \frac{(n-1)!}{n!}\mathbb{E}_1\left(I_{W_1}n^{U_1}\right) \\
&= \frac{1}{2}.
\end{aligned}$$

Hence

$$|\{B \in \mathcal{B}_n :\ W_1\}| = 2^{n-1} n!$$

and the result is obtained by combining this with (3.19) and (3.20). □

It follows from Lemma 25 that to prove Theorem 24, it suffices to determine the number of corners in type–B permutation tableaux of size n or the expected number of corners since $|\mathcal{B}_n| = 2^n n!$ and therefore equation (3.2) becomes

$$c(\mathcal{B}_n) = 2^n n!\, \mathbb{E}_n C_n. \tag{3.21}$$

Now consider the following preliminary theorem.

Theorem 26. *For type–B permutation tableaux of size n, the probability of having a corner with* *border edges k and $k+1$ is given by*

$$\mathbb{P}_n\left(I_{S_k,W_{k+1}}\right) \quad = \quad \frac{n-k+1}{2n} - \frac{(n-k)^2}{4n(n-1)}.$$

Proof. By (3.17),

$$\mathbb{E}_n\left(I_{S_k,W_{k+1}}\right) = \frac{2\cdot 3\cdot \cdots \cdot (n-k-1)}{n(n-1)\cdot \cdots \cdot (k+2)} \mathbb{E}_{k+1} I_{S_k,W_{k+1}} (n-k)^{U_{k+1}} \tag{3.22}$$

as obtained in (3.8). Now, consider

$$\mathbb{E}_{k+1} I_{S_k,W_{k+1}} (n-k)^{U_{k+1}} = \mathbb{E}_{k+1} I_{S_k} \mathbb{E}(I_{W_{k+1}} (n-k)^{U_{k+1}} | \mathcal{F}_k) \tag{3.23}$$

for $1 \le k \le n-1$. The conditional expectation is,

$$\mathbb{E}(I_{W_{k+1}}(n-k)^{U_{k+1}}|\mathcal{F}_k) = \mathbb{E}((n-k)^{U_{k+1}}|\mathcal{F}_k) - \mathbb{E}(I_{S_{k+1}}(n-k)^{U_{k+1}}|\mathcal{F}_k)$$

since the two indicators are complementary. The first conditional expectation on the right–hand side was computed in Theorem 21 (see (3.10)). To compute the second conditional expectation, note that on the set $\{S_{k+1}\}$, $U_{k+1} = 1 + U_k$ so that

$$\begin{aligned}
\mathbb{E}(I_{S_{k+1}}(n-k)^{U_{k+1}}|\mathcal{F}_k) &= (n-k)^{1+U_k}\mathbb{E}(I_{S_{k+1}}|\mathcal{F}_k) \\
&= (n-k)^{1+U_k}\mathbb{P}(S_{k+1}|\mathcal{F}_k) \\
&= (n-k)^{1+U_k}\frac{1}{2^{U_k+1}}
\end{aligned}$$

where the last equality follows from (3.15). Combining with (3.10) yields

$$\mathbb{E}(I_{W_{k+1}}(n-k)^{U_{k+1}}|\mathcal{F}_k) = (n-k)\left(\left(\frac{n-k+1}{2}\right)^{U_k} - \frac{1}{2}\left(\frac{n-k}{2}\right)^{U_k}\right)$$

and thus (3.23) equals

$$(n-k)\mathbb{E}_{k+1}\left(I_{S_k}\left(\left(\frac{n-k+1}{2}\right)^{U_k}-\frac{1}{2}\left(\frac{n-k}{2}\right)^{U_k}\right)\right).$$

The expression inside the expectation is a random variable on $\mathcal{P}_k$, therefore, (3.17) can be used to obtain

$$\frac{n-k}{k+1}\mathbb{E}_k I_{S_k}\left((n-k+1)^{U_k}-\frac{1}{2}(n-k)^{U_k}\right).$$

Furthermore, on the set $\{S_k\}$, $U_k = U_{k-1}+1$ so that the above is

$$\frac{n-k}{k+1}\mathbb{E}_k\left(\left((n-k+1)^{1+U_{k-1}}-\frac{1}{2}(n-k)^{1+U_{k-1}}\right)\mathbb{E}(I_{S_k}|\mathcal{F}_{k-1})\right),$$

which, by (3.15), equals

$$\frac{n-k}{k+1}\mathbb{E}_k\left(\left((n-k+1)^{1+U_{k-1}}-\frac{1}{2}(n-k)^{1+U_{k-1}}\right)\frac{1}{2^{U_{k-1}+1}}\right).$$

Reducing the size one more time gives

$$\frac{n-k}{2(k+1)k}\left(\mathbb{E}_{k-1}(n-k+1)^{1+U_{k-1}}-\frac{1}{2}\mathbb{E}_{k-1}(n-k)^{1+U_{k-1}}\right). \tag{3.24}$$

Combining (3.22) and (3.24) and applying (3.12) and (3.13),

$$\begin{aligned}\mathbb{E}_n\left(I_{S_k,W_{k+1}}\right) &= \frac{(n-k-1)!(k+1)!}{n!}\cdot\frac{n-k}{2k(k+1)}\left(\frac{(n-k+1)(n-1)!}{(k-1)!(n-k)!}-\frac{(n-k)(n-2)!}{2(k-1)!(n-k-1)!}\right)\\ &= \frac{n-k+1}{2n}-\frac{(n-k)^2}{4n(n-1)}.\end{aligned}$$

as desired. □

The relationship between permutation tableaux and tree–like tableaux given by (3.18) implies the following corollary to Theorem 26.

Corollary 27. *For symmetric tree–like tableaux of size $2n+1$, $n \geq 2$, the probability of having a* *corner with border edges k and $k+1$ is given by*

$$\mathbb{P}_n\left(I_{S_k,W_{k+1}}\right) = \begin{cases} \frac{1}{2n} & k=1 \\ \frac{k}{2n} - \frac{(k-1)^2}{4n(n-1)} & k=2,\dots n, \\ \frac{1}{2} & k=n+1 \\ \frac{2n-k+2}{2n} - \frac{(2n-k+1)^2}{4n(n-1)} & k=n+2,\dots 2n \\ \frac{1}{2n} & k=2n+1. \end{cases}$$

Finally, the following result, when combined with (3.18) and (3.21), completes the proof of Theorem 24.

Theorem 28. *For type–B permutation tableaux of size n,*

$$\mathbb{E}_n C_n = \frac{4n+7}{24} - \frac{1}{2n}.$$

Proof. As in the proof of Theorem 21 consider

$$\mathbb{E}_n\left(\sum_{k=1}^{n-1} I_{S_k,W_{k+1}}\right) = \sum_{k=1}^{n-1} \mathbb{E}_n\left(I_{S_k,W_{k+1}}\right).$$

Then the result is obtained by summing the expression from Theorem 26 from $k=1$ to $n-1$,

$$\begin{aligned} \mathbb{E}_n C_n &= \frac{1}{2}\sum_{k=1}^{n-1}\frac{n-k+1}{n} - \frac{1}{4}\sum_{k=1}^{n-1}\frac{(n-k)^2}{n(n-1)} = \frac{1}{2}\sum_{j=2}^{n}\frac{j}{n} - \frac{1}{4}\sum_{j=1}^{n-1}\frac{j^2}{n(n-1)} \\ &= \frac{n(n+1)}{4n} - \frac{1}{2n} - \frac{(n-1)n(2n-1)}{24n(n-1)} = \frac{4n+7}{24} - \frac{1}{2n} \end{aligned}$$

□

To conclude this section, note that Theorem 24 could also be obtained by summing the expression from Corollary 27 from $k=1$ to $2n+1$.

3.6 Occupied Corners in Tree–Like Tableaux

In this section, the asymptotic distribution of the number of occupied corners in random tree–like tableau and random symmetric tree–like tableau will be derived. In [31], Laborde Zubieta derived a recurrence for the generating polynomials for the number of occupied corners and used it to obtain the expected value and the variance of the number of occupied corners in tree–like tableaux. He also obtained similar results in the symmetric case. These results can be extended to identify the limiting distribution of the number of occupied corners in each of these two cases.

First, consider the following generalized statement.

Proposition 29. *Let $P_n(x) = \sum_{k=0}^{m} a_{n,k}x^k$ be a sequence of polynomials satisfying the recurrence*

$$P_n^{'}(x) = f_n P_{n-1}(x) + g_n(x-1)P_{n-1}^{'}(x) \tag{3.25}$$

for some sequences of constants (f_n) and (g_n). Assume that $a_{n,k} \geq 0$, $\sum_k a_{n,k} > 0$ for every $n \geq 1$, and that $m = m_n$ may depend on n, and consider a sequence of random variables X_n defined by

$$\mathbb{P}(X_n = k) = \frac{a_{n,k}}{P_n(1)} = \frac{a_{n,k}}{\sum_j a_{n,j}}.$$

If

$$g_n = o(f_n) \quad \text{and} \quad f_n \frac{P_{n-1}(1)}{P_n(1)} \to c > 0, \quad \text{as} \quad n \to \infty \tag{3.26}$$

then

$$X_n \xrightarrow{d} \mathrm{Pois}(c) \quad \text{as} \quad n \to \infty,$$

where Pois(c) *is a Poisson random variable with parameter $c > 0$.*

Proof. By [4 Theorem 20, Chapter 1] it is enough to show that for every $r \geq 1$ the factorial moments

$$\mathbb{E}(X_n)_r = \mathbb{E}X_n(X_n-1)\dots(X_n-(r-1)),$$

of (X_n) converge to c^r as $n \to \infty$. Recall that for a random variable X with generating function $h(x) = \mathbb{E}x^X$,

$$\mathbb{E}(X)_r = h^{(r)}(1),$$

where $h^{(r)}(x)$ is the r^{th} derivative of $h(x)$. Thus, if

$$\frac{P_n^{(r)}(1)}{P_n(1)} \to c^r, \quad \text{as} \quad n \to \infty$$

then the claim is proven.

Using (3.25),

$$\begin{aligned} P_n^{(r)}(x) &= \left(P_n^{'}(x)\right)^{(r-1)} = f_n P_{n-1}^{(r-1)}(x) + g_n \left((x-1)P_{n-1}^{'}(x)\right)^{(r-1)} \\ &= f_n P_{n-1}^{(r-1)}(x) + g_n \left((x-1)P_{n-1}^{(r)}(x) + \binom{r-1}{1} g_n P_{n-1}^{(r-1)}(x)\right) \end{aligned}$$

where in the last step Leibniz formula for the differentiation of the product of two functions was applied. It follows that

$$P_n^{(r)}(1) = \left(f_n + (r-1)g_n\right) P_{n-1}^{(r-1)}(1)$$

and, consequently,

$$\begin{aligned} \frac{P_n^{(r)}(1)}{P_n(1)} &= (f_n + (r-1)g_n) \frac{P_{n-1}^{(r-1)}(1)}{P_n(1)} \\ &= f_n \frac{P_{n-1}(1)}{P_n(1)} \left(1 + (r-1)\frac{g_n}{f_n}\right) \frac{P_{n-1}^{(r-1)}(1)}{P_{n-1}(1)}. \end{aligned}$$

Therefore,

$$\frac{P_n^{(r)}(1)}{P_n(1)} = \left(\prod_{k=0}^{r-1} f_{n-k} \frac{P_{n-k-1}(1)}{P_{n-k}(1)} \left(1 + (r-k-1)\frac{g_{n-k}}{f_{n-k}}\right)\right) \frac{P_{n-r}^{(r-r)}(1)}{P_{n-r}(1)}.$$

Since the last factor is 1, it follows from (3.26) that for every $r \geq 1$ as $n \to \infty$,

$$\frac{P_n^{(r)}(1)}{P_n(1)} \quad \to \quad c^r$$

as desired. □

For occupied corners, Laborde Zubieta obtained (3.25) with $f_n = n$ and $g_n = 1$. Since in that case $P_n(1) = n!$, the assumptions of Proposition 29 are clearly satisfied, with $c = 1$. Therefore, the limiting distribution follows immediately.

Corollary 30. *As $n \to \infty$, the limiting distribution of the number of occupied corners in a random tree–like tableau of size n is* Pois(1).

3.6.1 The Symmetric Case

For the symmetric tableaux of size $2n+1$, the generating polynomial of the number of occupied corners is

$$Q_n(x) = \sum_{k \geq 0} b_{n,k} x^{2k}$$

where

$$2k \cdot b_{n,k} = 2[2k \cdot b_{n-1,k} + (n - 2(k-1))b_{n-1,k-1}], \tag{3.27}$$

(see [31]). Set

$$R_n(z) = \sum_k b_{n,k} z^k, \quad \text{so that} \quad Q_n(x) = R_n(x^2).$$

Then (3.27) translates to

$$2zR_n^{'}(z) \quad = \quad 4zR_{n-1}^{'}(z) + 2nzR_{n-1}(z) - 4z^2 R_{n-1}^{'}(z).$$

Therefore,

$$R_n^{'}(z) = nR_{n-1}(z) + 2(1-z)R_{n-1}^{'}(z).$$

By Proposition 23, $R_n(1) = 2^n n!$. Thus, the conditions of Proposition 29 are satisfied with $f_n = n$, $g_n = -2$, and $c = 1/2$. That is, as $n \to \infty$,

$$\frac{R_n^{(r)}(1)}{R_n(1)} \quad \to \quad \left(\frac{1}{2}\right)^r.$$

Therefore, if Y_n is a random variable with the probability generating function $R_n(z)/R_n(1)$, then (Y_n) converges in distribution to a Pois(1/2) random variable. Moreover, $Q_n(x)/Q_n(1)$ is the probability generating function of $2Y_n$ which therefore, the limiting distribution in the symmetric case follows immediately.

Corollary 31. *As $n \to \infty$, the limiting distribution of the number of occupied corners in a random symmetric tree–like tableau of size $2n+1$ is $2 \times \mathrm{Pois}(1/2)$.*

3.7 Diagonal Cells in Symmetric Tree–Like Tableaux

In [1], it was shown that the expected number of diagonal cells in symmetric tableaux of size $2n+1$ is $3(n+1)/4$ (see [1 Proposition 19]). This section extends those results by proving that the number of diagonal cells is asymptotically normal. The precise statement is as follows.

Theorem 32. *Let D_n be the number of diagonal boxes in a random symmetric tree–like tableau of size $2n+1$ and $N(0,1)$ denote the standard normal random variable. Then, as $n \to \infty$*

$$\frac{D_n - 3(n+1)/4}{\sqrt{7(n+1)/48}} \xrightarrow{d} N(0,1).$$

This theorem will be proven by considering the generating polynomials for the number of diagonal cells in symmetric tree–like tableaux of size $2n+1$,

$$B_n(x) = \sum_{k=1}^{n+1} B(n,k)x^k, \quad n \geq 0,$$

(where $B(n,k)$ is the number of symmetric tree–like tableaux of size $2n+1$ with k diagonal cells).

In [1 Section 3.2], $B_n(x)$ was shown to satisfy the following recurrence,

$$B_n(x) = nx(x+1)B_{n-1}(x) + x(1-x^2)B'_{n-1}(x), \tag{3.28}$$

$$B_0(x) = x. \tag{3.29}$$

Note that (D_n) are random variables defined by

$$\mathbb{P}(D_n = k) = \frac{B(n,k)}{\sum_{k\geq 0} B(n,k)} = \frac{B(n,k)}{B_n(1)}.$$

If all roots of $B_n(x)$ are real then $B_n(x)$ can be written as a product of linear factors. Furthermore, since the coefficients are non–negative the roots are non–positive. Hence, these linear factors may be interpreted as the generating functions of $\{0,1\}$–valued random variables and then knowing that the variance of their sum converges to infinity suffices to conclude that the sum is asymptotically normal. More specifically, assume that

$$-\infty < \gamma_{i,n} \leq 0, i = 1, \dots, m$$

are roots of $B_n(x)$ and write $\pi_{i,n} = -\gamma_{i,n}$ so that $\pi_{i,n} \geq 0$. Then $B_n(x)$ has a factorization

$$B_n(x) = B(n,k) \prod_{k=1}^{m} (x + \pi_{k,n}),$$

so that

$$\mathbb{E}x^{D_n} = \frac{B_n(x)}{B_n(1)} = \prod_{k=1}^{m} \frac{x+\pi_{k,n}}{1+\pi_{k,n}} = \prod_{k=1}^{m} \left(\frac{x}{1+\pi_{k,n}} + \frac{\pi_{k,n}}{1+\pi_{k,n}} \right).$$

The factor on the right–hand side is the probability generating function of a random variable $\xi_{k,n}$ such that

$$\mathbb{P}(\xi_{k,n} = 1) = \frac{1}{1+\pi_{k,n}} \quad \text{and} \quad \mathbb{P}(\xi_{k,n} = 0) = \frac{\pi_{k,n}}{1+\pi_{k,n}}, \quad k = 1, \dots, m.$$

Moreover, since the product of the probability generating functions corresponds to taking sums of independent random variables we have that

$$D_n = \sum_{k=1}^{n} \xi_{k,n},$$

where $(\xi_{k,n})$ are independent. Therefore, it follows immediately from either Lindeberg or Lyapunov version of the central limit theorem (see e. g. [3 Theorem 27.2 or Theorem 27.3]) that

$$\frac{X_n - \mathbb{E}X_n}{\sqrt{\mathrm{var}(X_n)}} \xrightarrow{d} N(0,1),$$

as long as $\mathrm{var}(X_n) \longrightarrow \infty$ as $n \to \infty$.

Therefore, the theorem will be proved once it is shown that the variance of D_n is linear in n and that all roots of $B_n(x)$ are real. The precise statements are given in the two propositions below.

Proposition 33. *The variance of the number of diagonal cells in a random symmetric tree–like tableaux of size* $2n+1$ *is,*

$$\mathrm{var}(D_n) = \frac{7(n+1)}{48}. \tag{3.30}$$

Proposition 34. *For all* $n \geq 0$*, the polynomial* $B_n(x)$

a) has degree $n+1$ *with all coefficients non-negative, and*

b) all roots real and in the interval $[-1, 0]$.

Proof of Proposition 34. To prove a), proceed by induction on n. When $n = 1$,

$$\begin{aligned} B_1(x) &= x(x+1)\left(B_0(x) + (1-x)B_0'(x)\right) \\ &= x(x+1)\left(x + (1-x)\right) \\ &= x(x+1). \end{aligned}$$

Assume the statement holds for n, i.e. $B_n(x) = \sum_{k=1}^{n+1} a_k x^k$ for some coefficients a_k such that

$a_{n+1} > 0$ and $a_k \geq 0$ for all $1 \leq k \leq n$. Then,

$$\begin{aligned} B_{n+1}(x) &= x(x+1)\Big((n+1)B_n(x) + (1-x)B_n^{'}(x)\Big) \\ &= x(x+1)\left((n+1)\left(\sum_{k=1}^{n+1} a_k x^k\right) + (1-x)\left(\sum_{k=1}^{n+1} k \times a_k x^{k-1}\right)\right) \\ &= x(x+1)\left(\sum_{k=1}^{n}(n-k+1)a_k x^k + \sum_{k=1}^{n+1} k \times a_k x^{k-1}\right). \end{aligned}$$

Therefore, $\deg(B_{n+1}(x)) = n+2$ with all coefficients non-negative, which completes the proof of a).

The proof of part b) is an adaptation of an argument given in [19] and takes advantage of the fact that 0 and -1 are roots of $B_n(x)$ for every $n \geq 1$.

Notice that a) implies that $(B_{n+1}(x))$ has at most $n+2$ zeros. Denote the zeros of $B_n(x)$ in increasing order by $\gamma_{i,n}, i = 1, \ldots, m \leq n+1$ and proceed in a similar manner to the proof of [19 Theorem 5] by rewriting the recurrence as

$$\begin{aligned} B_{n+1}(x) &= \frac{x(1-x^2)}{K_n(x)}\left[\frac{n+1}{1-x}K_n(x)B_n(x) + K_n(x)B_n'(x)\right] \\ &= \frac{x(1-x^2)}{K_n(x)}\frac{d}{dx}\Big[K_n(x)B_n(x)\Big] \end{aligned}$$

where the second equality is valid provided that

$$\frac{n+1}{1-x}K_n(x) = K_n'(x), \text{ i. e. } \quad K_n(x) = (1-x)^{-n-1}.$$

Again, proceed by induction. When $n = 1$, $B_1(x) = x(x+1)$ and $\gamma_{1,1} = -1$ and $\gamma_{2,1} = 0$ are its zeros.

Now assume the result has been proven for $B_n(x)$ and that

$$-1 = \gamma_{1,n} < \gamma_{2,n} < \cdots < \gamma_{n+1,n} = 0$$

are its zeros. Since $K_n(x)B_n(x) = \frac{B_n(x)}{(1-x)^{n+1}} = 0$ at $\gamma_{1,n}, \dots, \gamma_{n+1,n}$ apply Rolle's theorem to

$$B_{n+1}(x) = x(1+x)(1-x)^{n+2}\frac{d}{dx}\Big[K_n(x)B_n(x)\Big]$$

on each of the intervals $[\gamma_{i,n}, \gamma_{i+1,n}]$, $i = 1, \dots, n$ to obtain roots, $x_1, \dots, x_{n-1}$ of $B_{n+1}(x)$ such that

$$-1 = \gamma_{1,n} < x_1 < \gamma_{2,n} < \cdots < x_{n-1} < \gamma_{n+1,n} = 0.$$

Since $B_{n+1}(-1) = 0$ and $B_{n+1}(0) = 0$,

$$-1 = \gamma_{1,n} = \gamma_{1,n+1} < \gamma_{2,n+1} < \gamma_{2,n} < \cdots < \gamma_{n+1,n+1} < \gamma_{n+2,n+1} = \gamma_{n+1,n} = 0.$$

□

Proof of Proposition 33. First the second factorial moment of D_n will be calculated. Differentiating the recurrence (3.29) twice and evaluating at $x = 1$ yields

$$B_n''(1) = 2nB_{n-1}(1) + 6(n-1)B_{n-1}'(1) + 2(n-2)B_{n-1}''(1).$$

Furthermore, since

$$B_n(1) = 2nB_{n-1}(1)$$

and

$$\operatorname{var}(D_n) = \mathbb{E}(D_n)_2 - \mathbb{E}^2 D_n + \mathbb{E}D_n, \tag{3.31}$$

$$\begin{aligned}
\mathbb{E}(D_n)_2 &= \frac{B_n''(1)}{B_n(1)} = \frac{2nB_{n-1}(1) + 6(n-1)B_{n-1}'(1) + 2(n-2)B_{n-1}''(1)}{2nB_{n-1}(1)} \\
&= 1 + \frac{3(n-1)}{n}\mathbb{E}D_{n-1} + \frac{n-2}{n}\mathbb{E}(D_{n-1})_2 \\
&= 1 + \frac{3(n-1)}{n}\mathbb{E}D_{n-1} + \frac{n-2}{n}\left(\operatorname{var}(D_{n-1}) + \mathbb{E}^2 D_{n-1} - \mathbb{E}D_{n-1}\right) \\
&= 1 + \frac{n-2}{n}\operatorname{var}(D_{n-1}) + \frac{n-2}{n}\mathbb{E}^2 D_{n-1} + \left(\frac{2n-1}{n}\right)\mathbb{E}D_{n-1}.
\end{aligned}$$

Now, using $\mathbb{E}D_n = 3(n+1)/4$ (as computed from (3.29) in [1 Proposition 19]) and (3.31) the following is obtained,

$$\begin{aligned}
\operatorname{var}(D_n) &= 1 + \frac{n-2}{n}\operatorname{var}(D_{n-1}) + \frac{n-2}{n}\left(\frac{3n}{4}\right)^2 + \frac{2n-1}{n}\frac{3n}{4} \\
&\quad - \left(\frac{3(n+1)}{4}\right)^2 + \frac{3(n+1)}{4} \\
&= \frac{n-2}{n}\operatorname{var}(D_{n-1}) + \frac{7}{16}.
\end{aligned}$$

This recurrence is easily solved using a technique from [22 Section 2.2]. First of all, express the recurrence in the following way,

$$n\operatorname{var}(D_n) = (n-2)\operatorname{var}(D_{n-1}) + \frac{7n}{16}.$$

Then multiply by a summation factor, $(n-1)$.

$$n(n-1)\operatorname{var}(D_n) = (n-1)(n-2)\operatorname{var}(D_{n-1}) + \frac{7n(n-1)}{16}.$$

Now define $S_n = n(n-1)\operatorname{var}(D_n)$. Then the recurrance can be expressed as

$$\begin{aligned}
S_n &= S_{n-1} + \frac{7n(n-1)}{16} \\
&= \sum_{k=1}^{n}\frac{7k(k-1)}{16} \\
&= \frac{7}{16}\left(\sum_{k=1}^{n}k^2 - \sum_{k=1}^{n}k\right) \\
&= \frac{7}{16}\left(\frac{n(n+1)(2n+1)}{6} - \frac{n(n+1)}{2}\right) \\
&= \frac{7n(n+1)(n-1)}{48}.
\end{aligned}$$

Now since $S_n = n(n-1)\operatorname{var}(D_n)$,

$$\operatorname{var}(D_n) = \frac{7(n+1)}{48},$$

which completes the proof of Proposition 33 and Theorem 32. □

Remark 1. *The representation of D_n as the sum of independent indicator random variables implies that a local limit theorem holds too. Specifically, using* $\mathbb{E}D_n = 3(n+1)/4$ *and* $\mathrm{var}(D_n) = 7(n+1)/48$ *it follows that*

$$\mathbb{P}(D_n = k) = \frac{2\sqrt{6}}{\sqrt{7\pi(n+1)}} \exp\left(-\frac{24(k-3(n+1)/4)^2}{7(n+1)} + o(1)\right)$$

holds uniformly over k as $n \to \infty$. See [23 Theorem 2.7 and a discussion of its proof in Section 5] for more detailed explanation and [36 Theorem VII.3] for a general statement of a local limit theorem.

Chapter 4: Conclusion

The results from this thesis confirm several conjectures on tableaux in the literature which are significant in terms of the ASEP. First, the distribution of the kth diagonal of staircase tableaux, where $k \geq 2$ is fixed, was proven to be asymptotically Poisson partially confirming a conjecture of Hitczenko and Janson [23]. This result relates to the ASEP by the fact that the first diagonal of staircase tableaux dictates the state of the ASEP to which it is associated. Second, the number of corners in tree–like tableaux and symmetric tree–like tableaux were determined, confirming conjectures of Laborde-Zubieta [31]. Corners in tree–like tableaux correspond to places in the PASEP where particles have the opportunity to move.

This thesis also established the groundwork for expected value calculations on type–B permutation tableaux in the way of Corteel and Hitczenko for permutation tableaux [8]. Using those results, the expected number of corners in permutation tableaux and type–B permutation tableaux were obtained. These calculations were motivated by the conjectures of Laborde-Zubieta [31] and the bijections mentioned in [1].

Lastly, several results from this thesis extend previous results on staircase tableaux and tree–like tableaux. First, the distribution of boxes on the kth diagonal of staircase tableaux, where $k \geq 2$ is fixed, was computed which extends results of Hitczenko and Janson on the first diagonal [23]. In addition, the limiting distribution of the number of occupied corners in tree–like tableaux was derived extending results of Laborde-Zubieta [31]. Finally, the limiting distribution of the number of diagonal cells was derived, extending results of Aval et. al [1].

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Vita

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